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Stochastic Population Dynamics for Regional Water Supply and Waste Management Decision-Making

Peter Meier

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ENVIRONMENTAL ENGINEERING
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STOCHASTIC POPULATION DYNAMICS FOR REGIONAL WATER SUPPLY AND WASTE MANAGEMENT DECISION-MAKING

by

Peter Meier
Ph.D.

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ABSTRACT

A rational methodology for local area population projection and water and sewer service area prediction is developed. The projection model consists of a stochastic simulation of inter-regional population growth and a finite-difference solution to a non-linear differential equation describing spatial variations in urban population densities. The projection model output is designed as input to optimization algorithms for regional water supply and waste treatment facilities. The components of demographic change are modeled as autoregressive stochastic processes, and a response surface algorithm is developed to decompose net migration rates into in- and outmigration rates. Service area prediction is based on a computerized evaluation of the distance-density relations at the existing service area periphery. Comparison of results to preliminary 1970 census figures indicates a superior prediction performance over traditional methods of population projection as practiced by consulting engineers and planners.

VITAE

The writer was born in Beaconsfield, England in 1942 and attended St.Paul's School, Hampstead and The Haberdashers Askes School, Elstree. He graduated from the Swiss Federal Institute of Technology Zurich, in 1966 with the degree of Dipl.Natwiss.ETH, with specialization in geography and regional planning. After attending the Sanitary and Water Resources Engineering Program at Vanderbilt University, Nashville, Tenn., the writer transferred to the Environmental Engineering Program at the University of Massachusetts in 1967. On graduation in 1968 with the M.Sc. in Civil Engineering, he participated in the Advanced Waste Treatment Evaluation Project of the Massachusetts Water Resources Research Center. On graduation the writer will commence on a Post-doctoral Fellowship at the Institut fur Siedlungswässerwirtschaft, University of Karlsruhe, Germany,and will serve as systems consultant to the Engineering and Planning firm of Curran Associates, Northampton, Mass.

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NOTATION*

α = Intercept term of linear regression models

b_i^t = Number of births in period t, region i [M]

β_i = birth rate, region i $[T]^{-1}$

B = diagonal matrix of birth rates

d_i^t = number of deaths in period t, region i [M]

δ_i = death rate, region i $[T]^{-1}$

Δ = diagonal matrix of death rates

e_k = (k x 1) unit vector

f = amplification factor

Γ = interregional growth operator

γ_i = i-th row of the interregional growth operator Γ

h = population density $[M]/[L]^2$

h_o = extrapolated population density at the urban centre, $[M]/[L]^2$

h_{ij}^t = population density at cartesian coordinates i,j at time t

I = identity matrix

K_x = permeability, x-direction $[L]^4/[T][M]$

K_y = permeability, y-direction $[L]^4/[T][M]$

K^* = intrinsic permeability $[L]^2/[M]$

$\ell_i^t(k)$ = number of net migrants in the k-year period commencing at time t, region i [M]

λ_i = net migration rate, region i

m_i^t = number of total inmigrants to region i, period t [M]

μ_i = inmigration rate, region i

(*) [M],[L],[T] represent physical dimensions as used in Chapter VIII

μ_{ij} = place specific migration rate from region i to region j,

M = matrix of place-specific migration rates μ_{ij}

N = sample size

n = length of intercensal interval, years [T]

o_i^t = number of outmigrants, period t, region i [M]

ω_i = outmigration rate, region i $[T]^{-1}$

Ω = diagonal matrix of outmigration rates ω_i

P_{ij} = fraction of individuals in j-th age group, region i [M]

q_i^t = growth increment, period t, region i [M]

ϕ_i^t = growth rate, period t, region i

ψ = mobility factor $[L]^2/[T]$

Φ = matrix of survival ratios

ρ = first order serial correlation coefficient

R = matrix of first-order serial correlation coefficients

$\rho_i(\ )$ = first-order serial correlation coefficient of the random variable ( ), region i

R_s = ratio of base populations

r = distance to the city (well) center [L]

r_w = well radius (=radius of CBD) [L]

r_e = equilibrium radius [L]

R_p^t = population projection range, period t

$\bar{R}_p^t$ = relative population projection range, period t

$\bar{s}_i(\ )$ = sample standard deviation of the random variable ( ), region i

u = migration velocity, x-direction [L]/[T] (Chapter VIII)

v = migration velocity, y-direction [L]/[T] (Chapter VIII)

u^t = vector of error terms, period t

v = random normal deviate

V = matrix of random normal deviates

w_i^t = population of region i, period t

w^t = vector of populations at time t

W = interregional population record

W_K = block-diagonal matrix of blocks W

x_i^t = population projection for region i, period t

X_{ji} = j-th explanatory variable, region i

y_j = lagged j-th column of the interregional population record W

z_i = segment of population not susceptible to migration, region i

A = estimate of A

$|A|$ = determinant of the square matrix A

$||A||$ = norm of the matrix A

Cond(A) = condition number of the matrix A

$\otimes$ = element by element multiplication of matrices

$\ominus$ = element by element division of matrices

$C \sim N(\mu,\sigma^2)$ = C is distributed normally with mean μ and variance σ^2

[L] = dimension of length

[T] = dimension of time

[M] = dimension of mass(individuals)

LIST OF ABBREVIATIONS

CBD	=	Central Business District
CC	=	Central city
c.of g.	=	center of gravity
E{ }	=	expectation of the random variable { }
LPVRPC	=	Lower Pioneer Valley Regional Planning Commission.
LPVRPD	=	Lower Pioneer Valley Regional Planning District
LS	=	least squares
MAD	=	minimum absolute deviations
plim	=	probability limit
RND	=	random normal deviate
r.v.	=	random variable
SEA	=	State Economic Area
SF	=	single family (housing)
SMSA	=	Standard Metropolitan Statistical Area
ULS	=	unrestricted least squares
Var{ }	=	variance of the random variable { }

CHAPTER I

THE PROBLEM STATEMENT

Introduction

In recent years, the regional approach to waste management has received increasing attention from economists, location theorists, students of government, industrial engineers and applied mathematicians as well as from the environmental engineering profession itself. This has followed recognition of the scale economies of large regional treatment plant facilities over several independently operated small plants and the intangible benefits of a unified regional waste treatment authority, both of not inconsiderable importance in view of contemporaneous pressures to establish efficient pollution control procedures and public concern over rising governmental expenditures. As the societal goal of environmental quality assumes major political significance, reflected in the legislation of upgraded water and air quality standards and concomitant jurisdictional enforcement powers, so will waste treatment facilities, and their operation, become ever more complex. Universal secondary treatment of municipal wastes can be anticipated within the next decade under the powers of the proliferation of State and Federal Environmental Quality Acts. Advanced waste treatment will become necessary to meet effluent standards imposed on large population agglomerations where the sheer magnitude of municipal wastes demands a superior contaminant removal performance than elsewhere necessary,[1] in addition to situations where water reclamation for industrial and municipal re-use is envisaged. Recent wotk by Smith (2) has again underscored the economies of scale that exist for the capital costs of primary, secondary and tertiary treatment facilities, and the complexities of efficient operation and maintenance of secondary and tertiary treatment clearly require large regional plants of sufficient size to support qualified operating personnel. A further argument for regionalisation that has to date recieved insufficient attention is the superior reliability of population forecasts for larger regions. This point will be elaborated in some detail in Chapter VII.

Such a large waste-treatment complex may be viewed as a system of subsystems comprising the elemental sewage sources, collection, regional interceptor, storage, treatment plant and recieving stream subsystems as shown in Figure 1. Each of these subsystems may itself consist of a number of subsystems. For example, the treatment plant subsystem consists of a number of interdependent biological and physical processes.

A convenient conceptual framework applicable to such a scheme centers around the notion of the "black box". For the purposes

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[1] See e.g. Lynam et al. (1), who report on the experiments at Metro Chicago to evaluate advanced treatment methods for the purpose of meeting the intended upgraded effluent standards of the next decade.

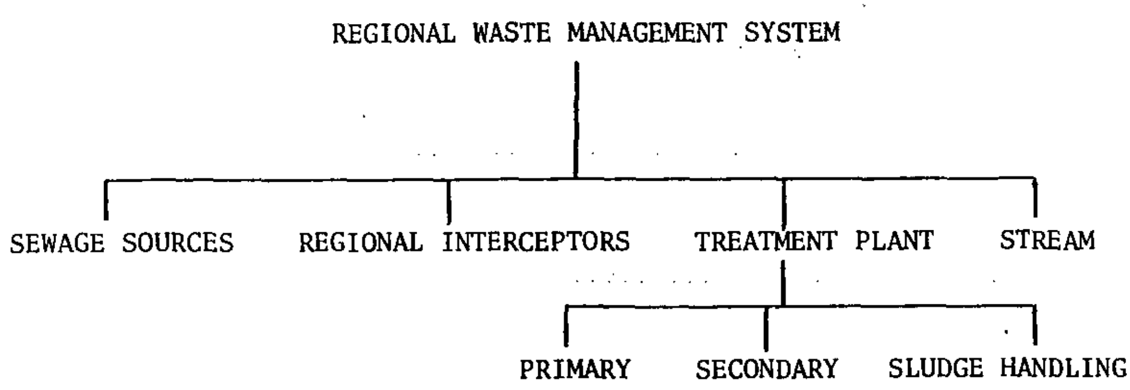

Figure 1 : The hierarchy of systems

of this study, this black box will comprise the regional interceptors, storage and treatment facilities and that section of the receiving stream utilised for waste assimilative purposes as illustrated in Figure 2. These subsystems are generally of a physical-engineering nature, amenable to mathematical modeling and optimization using modern tools of operations research. Considerable progress has been attained in the last decade in the quantitative formulation of the component systems of the treatment plant subsystem.[2]

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[2]For example, the trickling filter has been modeled by Swilley and Atkinson (3) and Galler and Gotaas (4), the activated sludge process and its modifications by Grieves et al. (5), and Erikson and Fan (6), the digestor by Pfeffer (7), and sludge drying by Nebiker et al. (8), and Meier et al. (9).

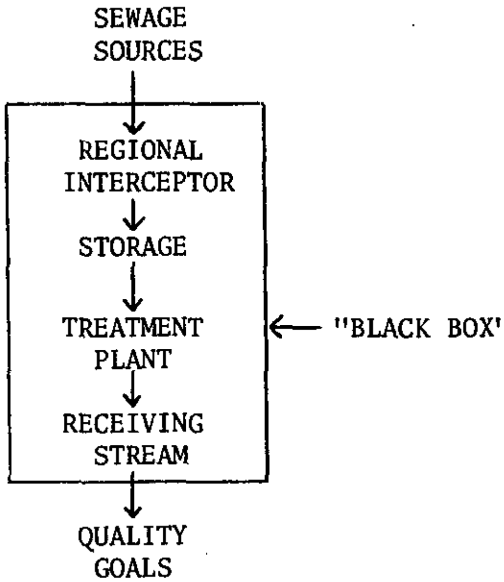

Figure 2 : The "black box" concept

Models of the treatment plant subsystem itself have made use of simulation and linear programming techniques[3] and on-going research is attempting the necessary developments, refinements and extensions necessary for regional optimization purposes.[4]

Specification of the black box output has attracted the particular attention of economists, concerned with the economic efficiency considerations resulting from the imposition of stream standards and the necessary waste treatment costs incurred in their successful attainment.[5]

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[3]For example Montgomery and Lynn (10), Shih and DePilippi (11), and Lynn et al.(12)

[4]See e.g. Giglio et al.(13) and Adrian (14)

[5]In the wake of Kneese's work in the early sixties have followed numerous studies consemed with regional water quality models (see e.g. Revelle et al. (15), Liebman and Lynn (16), and Roger and Gemmell (17)).

However, consideration of the input to the black box has to date been neglected, and it is to this problem that this study will be directed. Indeed, only one paper has been encountered in the sanitary engineering literature of the last decade that has focused attention on population projection methodology and it was essentially a review.[6]

The inputs to this black box are the "sewage sources". These are the locations at which wastewater is generated, and may originate from participant communities, residential developments, industrial zones etc. Associated with each source is a set of pertinent quality and quantity characteristics, denoted "stream vectors".[7] Since the optimization step includes consideration of the spatial location of constituent subsystems, it is evident that such sewage sources must be specified in space as well as time. Given, for example, the assertion that the quantity of domestic sewage is primarily a function of population size, then the intra-regional population distribution assumes co-equal importance to the overall total regional population. This follows directly from the necessity of efficient sizing and location of regional interceptors.

It becomes evident that in the specification of the system input, the dominance of physical-engineering considerations is superseded by the socio-economic forces that govern population growth and

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[6] McJunkin (18)
[7] Following the notation of Smith (19)

distribution, industrial location and political organisation. Indeed, of immediate concern is the selection of the planning region itself. On what basis should a particular community be included or excluded from the black box optimization? In view of the infinity of permutations of regional associations, some prior limitation is essential if the black box is to be kept to within reasonable dimensions.

Just as the optimization step requires a successful synthesis between the environmental engineering disciplines and operations research, so will an adequate treatment of the system input demand a synthesis of the environmental and social sciences, and, in particular, demography and the regional sciences. It is thus the purpose of this study to attempt such an interdisciplinary synthesis for the specific requirements of regional waste management planning models, and to develop a methodology for the formualtion of the input to regional optimization procedures.

The Problem Statement

Delineation of the problem. Specification of the black-box input falls into three logical phases. The first is to estimate the anticipated future population of the region under consideration. The second is an evaluation of the extent of the future sewer service area. In view of the dependence of domestic sewage flows on residential water consumption, a simultaneous consideration of the water service area will be necessary. The third step is the transformation of the serviced population into the desired stream vector - the expected flows, and quality factors.

This study is restricted to the first two of the aforementioned steps in view of the interrelationships that exist between the spatial location of residential developments and the availability of municipal services,of which access to the sewage collection system and public water supply are unquestionably dominant as locational determinants.

A further restriction is the focus on residential location. Location of central place services and employment are not considered explicitly, and the resulting projections are designed for the purpose of providing a rational basis for estimating domestic sewage flows and water demands, to the exclusion of commercial-industrial wastewaters.

Principal focus of the study. The most serious shortcomings of existing techniques available to develop the required stream vectors are the deterministic nature of local area population projection methods and the inherent subjectivities of estimating a future service area. Modern capacity-expansion optimization algorithms are not restricted to the single time period deterministic demand functions traditional to the design of treatment facilities by the environmental engineering profession. Recent interest in optimal time-capacity expansion of wastewater treatment systems (Thomas (20), Rachford *et al.* (21) and Scarato (22)), although presently restricted to linearly increasing demand functions, point clearly to future developments. Results for more realistic demand patterns (geometric and arbitrary non-decreasing) are, however, readily available in the operations research literature.[8] The dependence of the timing of capacity expansions on the interest rate, time horizon and demand variability has there long been established.

The economies of scale of regional treatment facilities are offset by the cost of regional interceptors, thus limiting potential regional facilities to relatively small areas. Unfortunately the sophisticated tools of mathematical demography available for the analysis of closed population systems are unsuited to local area population projections. Consequently, local area projections have gained notoriety as being extremely unreliable, in that the relatively primitive deterministic extrapolations that are still in widespread

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[8]See for example Veinott and Manne (23) and Srinivasan (24)

use by both the planning and engineering professions yield quite inaccurate results. The detailed review of present practice of the following section will indicate the extent to which such methods still persist in the professions involved in the realities of the regional planning process.

Review of Present Population Projection Practice

Local area projections by planning consultants. The population projections prepared for the Lower Pioneer Valley Regional Planning Commission (LPVRPC) by their planning consultants fully reflect the aforementioned inadequacies of the existing projection methodology for small areas. Two methods were employed; straight line projections, based on a least squares fit over the interval 1910-1960 or 1940-1960, and the step-down method. For the total study area, the high projection was derived from the step-down method[9] and the low from the straight-line projection. Community projections also consisted of high and low estimates, again using the above methods, modified to some extent by judgements based on evaluation of local conditions. It would appear that the use of two different techniques to obtain high and low projections contradicts fundamental concepts of logical consistency. Furthermore, although the variability of a projection is recognized, there is no attempt to quantify this variability objectively. The best that can be said for such projections is that they are presented with

[9] For a description of the most-encountered current population projection techniques see Appendix D.

due warning as to their use, and that much effort is expended in arriving at a set of presentable numbers.

Examination of projections by other planning consultants indicates that the practice of using two different deterministic methods to derive high and low estimates is widespread.[10] Indeed, it will be shown in Chapter VII that the prediction performance of these deterministic projections are consistently inferior to their stochastic counterparts that will be developed in the course of this study.

Population projection by environmental engineers. Some indication of the state of the art of population projection in the environmental engineering field is the treatment of the topic in the latest texts. The new edition of "Water and Wastewater Engineering" by Fair, Geyer and Okun (25) includes extensive chapters on optimization techniques and stochastic hydrology, yet devotes only four of approximately 1,000 pages to population projections, and quotes but two references dating to 1940 and 1952. Mathematical curve fitting and subjective graphical extrapolation are the only methods elaborated. The authors conclude:

> "Plots of population against time generally exhibit trends that can be carried forward to the end of design periods. The eye of a skilled interpreter of population growth will guide his hand to extend population curves into what appear to be reasonable forecasts without committing the forecaster to a particular mathematical system. For this reason, graphical forecasts are much used by engineers." (op.cit.p.5-9)

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[10] For example, the Pittsfield Urbanized Area Transportation study used the cohort survival method as the "high", and the step-down method for the "low" projection. The population projections for the Franklin County Regional Planning Commission utilized the cohort-survival method for the low and an employment forecast related method for the high projection.

Such is the nature of the offerings on population projection in a modern text, widely acknowledged as an authorotative work. Figures 3 and 4 demonstrate how such forecasts turn out in practice. Both are totally unrealistic, upon which considerable expenditure for capacity expansion is justified, and both are quite typical of projections by engineering consultants. They characterize a deeply ingrained design philosophy that has persited to the present day in the absence of a demonstrably superior alternative.

Relevancy of the Study

From the foregoing review it is evident that there has been a lack of response from the academic community to the needs of the real world. The professions cannot be indicted for a continued reliance on poor techniques while superior alternatives are still lacking. With rapidly increasing public monies expended on environmental pollution control to attain goals that must compete with the complex socio-economic problems of the domestic sector for priority, a re-evaluation of the projection basis upon which treatment facilities are planned and constructed is urgently required. Recent research efforts have been directed principally toward the black box itself, neglecting the input upon which the results must rest. Specification of design capacity is unquestionably the single most important factor in treatment plant design, a specification that has traditionally rested on a projection methodology of questionable objectivity.

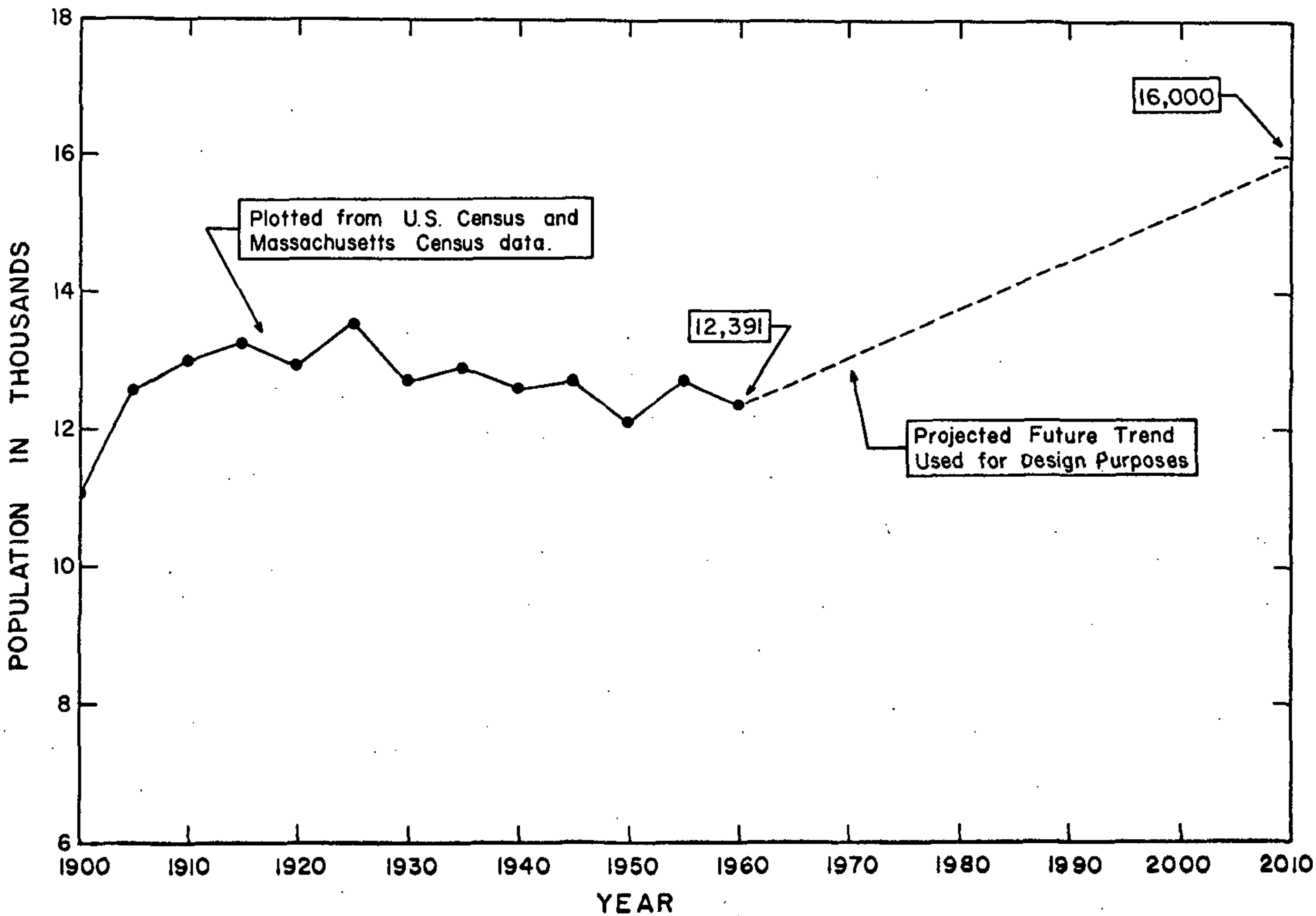

Figure 4 : Population projection by an engineering consultant for determination of sewage treatment plant design capacity for a small industrial town in Western Massachusetts.

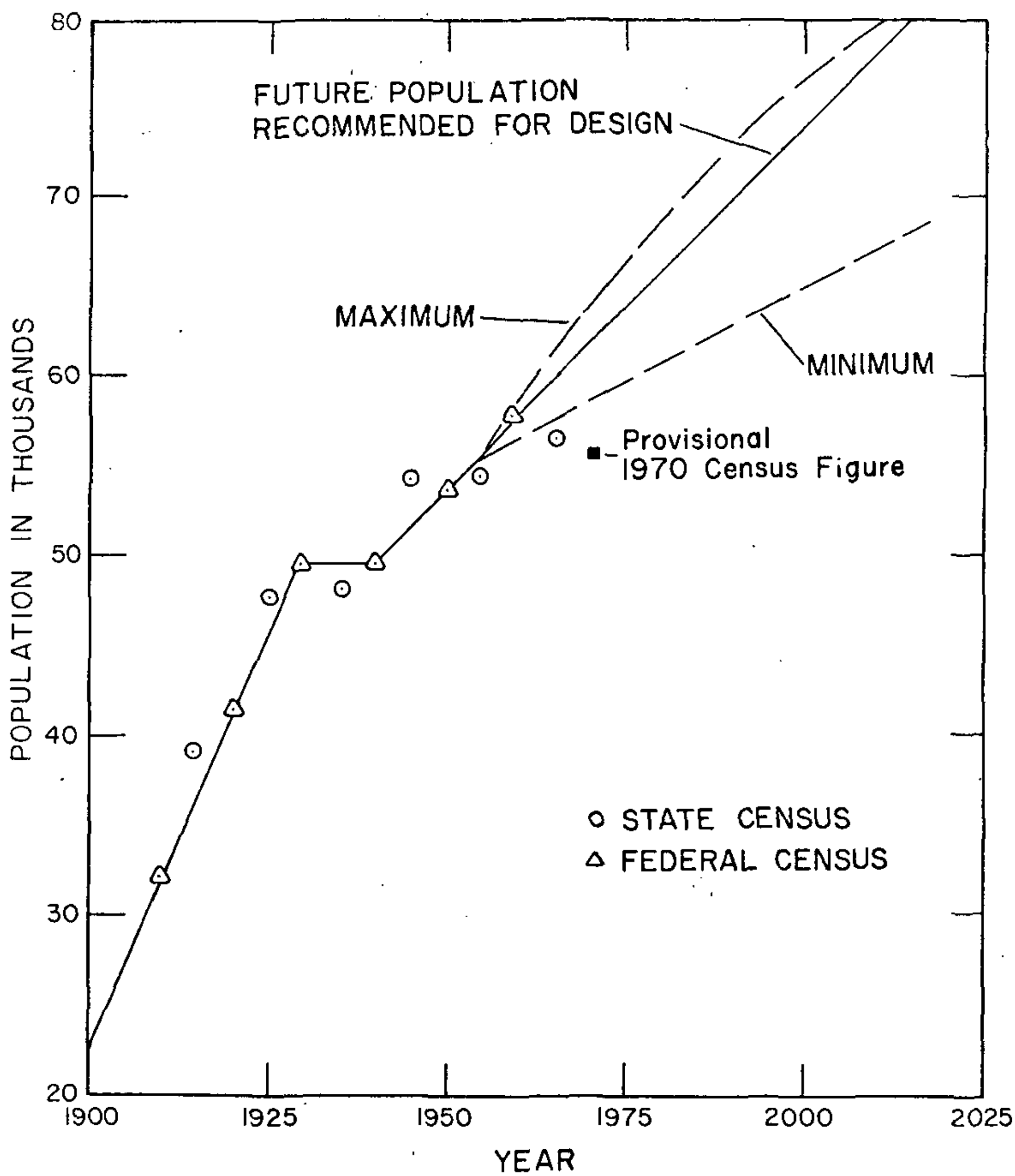

Figure 3 : Population by an engineering consultant for a water demand forecast for a New England city to justify additional source development. Note that the minimum projection for 1965 lies above the 1965 census figure, which was available at the time of the projection.

This study will therefore attempt to develop new approaches to a hitherto neglected field, relevant both in terms of the current research effort toward quantitative analysis of environmental problems and in terms of immediate priorities for the planning professions.

CHAPTER II

APPROACHES TO PROBLEM RESOLUTION

The present availability of time-sharing computer facilities no longer precludes practical utilization of computerized mathematical techniques by the consulting engineer. The models to be developed in this study thus rest heavily on numerical methods and stochastic simulation techniques that require computer facilities for implementation. The Lower Pioneer Valley Regional Planning District (that includes the Springfield-Holyoke-Chicopee urbanized area is used throughout for sample calculations.

Data base. Sample calcultations to illustrate projection models and their soluation algorithms utilize 1965 as a base year, and their short-term performance is measured against preliminary 1970 census figures available at the time of writing. The analysis of migration rates utilizes 1950-1960 census data; the relationships established need be updated on availability of detailed 1970 census results. Although this is unfortunate from the point of view of presenting definitive projections for the study region, it is unavoidable for any project conducted toward the end of an intercensal interval. The reader is reminded that the prime purpose of this study is the generalized development of new techniques rather than a presentation of complete results for a specific region

Outline of the Study

Population projection. Attention is first focused on migration, the major component of local area population change. The service area prediction model requires that net migration be decomposed into in- and outmigration streams, data that is not generally available. A model developed by Rogers (26) is conceptually attractive for this decomposition, but computational experience to date indicates poor numerical performance as measured against prior knowledge of migration behaviour. Chapter III will introduce this model and analyze the reasons for its unsatisfactory performance. The concept of migration rates as serially dependent random variables is introduced and developed in Chapter IV. Chapter V develops a numerical solution algorithm for the migration decomposition that utilizes a response surface minimization to evaluate objectively the optimum degree of data smoothing required to eliminate the errors of intercensal population estimation.

Births and deaths are modeled as autoregressive stochastic processes in Chapter VI and Chapter VII integrates the components of demographic change into a viable projection technique. A stochastic simulation model provides the necessary mathematical-statistical framework. Output is in the form of a probability distribution, from which the projection uncertainties can be objectively evaluated.

<u>Service area prediction</u>. On the basis of a physical analogy to a well-known model of a recharge well, a non-linear differential equation is derived to describe spatial variations in residential population densities, utilizing a finite difference algorithm for solution (Chapter VIII). The resultant distance-density profiles are shown to be consistent with empirical formulae developed by urban geographers. Using the output of the population projection as a driving force, the model will simulate the future spatial distribution of residential location. Chapter IX considers the criteria for expansion of the water and sewer service areas and presents the computerized computational model that predicts the requisite serviced populations.

CHAPTER III

THE ROGERS MODEL FOR THE ESTIMATION OF INTERREGIONAL MIGRATION RATES

Review

Notation. Except where explicitly noted, lower case Greek letters will be used for rates (birth, death, migration rates etc.), and lower case letters for events (number of births, deaths, migrations). Subscripted lower case letters denote vectors, and double subscripted lower case letters denote scalars.

Upper case letters will represent matrices. The dimensions of matrices and matrix equations are indicated by the notation (i x j) immediately below the corresponding matrix expression. The usual rules of matrix algebra apply throughout. However, element by element multiplication of matrices will be denoted by the symbol $\otimes$ and element by element division by $\ominus$. The superscript T indicates matrix transposition, and the superscript -1 indicates inversion. Exponentiation is represented by the notation exp( ) or by a bracketed superscript.

The components-of-change model. The aforementioned notational rules are best introduced by consideration of the basic components-of-change model of population dynamics, namely

$$w_i^{t+1} = w_i^t + b_i^t + m_i^t - d_i^t - o_i^t$$

where w_i^t = population of region i at time t

b_i^t = number of births in region i between time t and t+1

d_i^t = number of deaths in region i between time t and t+1

m_i^t = number of inmigrants to region i between t and t+1

o_i^t = number of outmigrants from region i between t and t+1

Expressing Eq.[3.1] in terms of crude rates

$$w_i^{t+1} = w_i^t \, (1 + \beta_{ii} - \delta_{ii} + \mu_{ii} - \omega_{ii}) \quad \ldots \ldots \quad [\ 3.2\ ]$$

where β_{ii} = crude birth rate for region i

δ_{ii} = crude death rate for region i

μ_{ii} = crude inmigration rate for region i

ω_{ii} = crude outmigration rate for region i

Eq.[3.2] can be written as

$$w^{t+1} = w_i^t \, \gamma_{ii} \quad \ldots \ldots \ldots \ldots \quad [\ 3.3\ ]$$

where γ_{ii} is defined as the growth multiplier.

Interregional formulation. Consider now the application of the components of change formulation of population growth to an interregional population system of k regions, namely

$$\underset{(k \times 1)}{w^{t+1}} = [\ \underset{(k \times k)}{I} + \underset{(k \times k)}{B} - \underset{(k \times k)}{\Delta} + \underset{(k \times k)}{M} - \underset{(k \times k)}{\Omega}\ ] \underset{(k \times 1)}{w^{t}} \quad [\ 3.4\ ]$$

$$\underset{(k \times 1)}{w^{t+1}} = \underset{(k \times k)}{\Gamma} \; \underset{(k \times 1)}{w^{t}} \quad \ldots \ldots \ldots \ldots \quad [\ 3.5\ ]$$

where

w^t = (k x 1) vector whose i-th element denotes the population of region i at time t

I = identity matrix

B, Δ, Ω = diagonal matrices of elements β_{ii}, δ_{ii}, ω_{ii} respectively

M = matrix of place specific migration rates.

Γ = interregional growth operator

The elements of Γ are denoted as

$$\gamma_{ij} \text{ for } i = j$$

$$\mu_{ij} \text{ for } i \neq j$$

and from Eq.[3.5] it is clear that

$$\gamma_{ii} = 1 + \beta_{ii} - \delta_{ii} - \omega_{ii}$$

$$\mu_{ij} = \text{migration rate from region } j \text{ to region } i$$

$$M = \Gamma \text{ for all } i \neq j$$

Also we may define an (n x k) matrix W, the k-regional population record over an n-year time interval, such that

$$\underset{(n \times k)}{W} = \begin{bmatrix} w_1^t & w_2^t & . & . & . & w_k^t \\ w_1^{t+1} & w_2^{t+1} & . & . & . & w_k^{t+1} \\ . & & & & & \\ . & & & & & \\ w_1^{t+n-1} & w_2^{t+n-1} & . & . & . & w_k^{t+n-1} \end{bmatrix}$$

and an (n x 1) vector y_j that represents the j-th column of W lagged by one time period. Thus

$$\underset{(n \times 1)}{y_2} = \begin{bmatrix} w_2^{t+1} \\ w_2^{t+2} \\ . \\ w_2^{t+n} \end{bmatrix}$$

The difference between y_j, representing the lagged j-th column of W, and w^t, representing the t-th row of W, should be fully noted.

<u>The Rogers Model</u>. Rogers (26) proposed a method for estimating the interregional growth operator from the interregional population distribution by application of least squares to the relation

$$y_i = W \gamma_i \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [\ 3.6\ ]$$

where γ_i is the transpose of the i-th row of Γ, and for which the least squares estimate of ${}_i$ is given by

$$\hat{\gamma}_i = (W^T W)^{-1} W^T y_i \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [\ 3.7\ ]$$

which is repeated for all k rows to yield estimates for all the elements of Γ. Rogers (26) found that unrestricted least squares (ULS) estimates of Γ frequently contained negative elements in the off-diagonal elements which are by definition unacceptable. Although results obtained by a minimum absolute deviations (MAD) estimator bypassed this problem, the other major defect of estimated coefficients being of an unlikely magnitude was not eliminated. For example, the 1950-1960 time series of the two region system California - United States yielded the following estimate of Γ:

$$\hat{\Gamma} = \begin{bmatrix} 1.0054 & 0.0030 \\ 0.0808 & 1.0080 \end{bmatrix}$$

for which the off-diagonal elements overestimate observed migration flows. Simple smoothing schemes did not rectify this deficiency.

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[1] The minimum absolute deviations estimator is derived in Appendix B. This estimator utilizes a linear programming formulation and the Simplex algorithm, and negative coefficients do not occur.

Assumptions of the Rogers Model. The principal assumption underlying an unbiased least squares estimation of the rows of the interregional growth operator from the estimating equation [3.6] is that the γ_i (and hence also Γ) are indeed constants. Since it is quite unrealistic to assume an error free population record, Eq.[3.6] is required to be written as

$$y_i = W\,\gamma_i + u^t \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [\ 3.8\ ]$$

where u^t is a vector of error terms. For least squares estimates to be unbiased, we further require the error terms u^t to be distributed with zero mean. If an unadjusted intercensal population estimate is used for W, this condition is by no means assured. More serious, however, in view of the autoregressive nature of Eq.[3.8].(since y_i is a lagged column of W), is the fact that the errors may be serially correlated. Under such circumstances it is known that the least squares estimates of γ_i are seriously biased.

The purpose of the Monte Carlo studies of the following sections is to examine quantitatively the effect of various assumptions about the error term on the estimation results, and to suggest certain modifications to the estimating procedure such that the realities of demographical data are more fully considered.

In Chapter III we shall assume that the error term u^t is introduced by faulty specification of the intercensal population record; i.e. that births, deaths and migration movements are incorrectly recorded (enumeration error). In Chapter IV we shall examine the case for which additive error consists of both enumeration error and error introduced by virtue of migrations to and from regions not included in the system from which the growth operator elements are being estimated (specification error). Finally we shall consider the case for which the interregional growth operator elements are no longer constants but random variables.

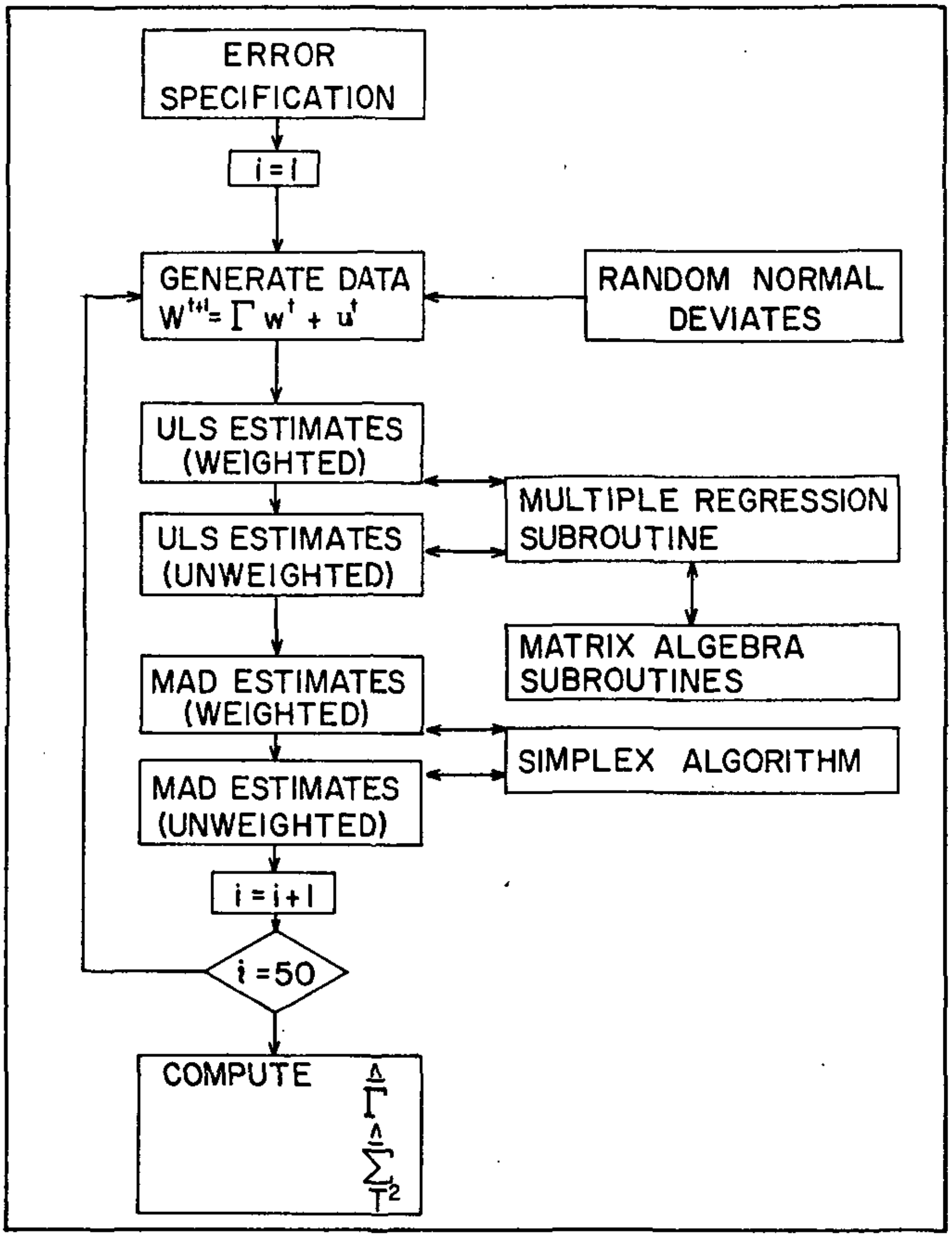

Figure 5 : Flow chart, Rogers Model Monte Carlo study (PROGRAM MCM)

Monte Carlo Evaluation of the Rogers Model

The estimates of interregional migration rates obtained by Rogers for real population series cannot be described as realistic. Although Rogers has published the results of only a limited number of actual computations, the model appeared to possess sufficient potential to warrant a systematic statistical study. To this end a Monte Carlo simulation was initiated. Data was generated artificially, using an assumed set of migration and vital rates to obtain the interregional population record W. The methods of least squares and minimum absolute deviations were then applied to these data, and the resultant estimates of the interregional growth operator compared to the true value of Γ used in generating the data.

The data were generated by sequential application of the equation

$$w^t = \Gamma\, w^{t-1} + u^t \quad \text{[3.9]}$$

where u^t is a vector of random disturbances. Suppose N data sets are generated, and let W_j be the j-th data record so generated. The j-th estimate of γ_i denoted $\hat{\gamma}_{ij}$, is given by

$$\hat{\gamma}_{ij} = (W_j^T W)^{-1} W_j^T\, y_{ij} \quad \text{[3.10]}$$

which is a random variable since W_j (and hence y_{ij}) are random variables. Further let

$$\hat{\hat{\gamma}}_i = \frac{1}{N} \sum_{j=1}^{N} \hat{\gamma}_{ij} \quad \text{[3.11]}$$

be the sample mean of the N estimates $\hat{\gamma}_{ij}$. Now if $u^t = 0$ for all t, then $W_1 = W_2 = \ldots = W_N$ and hence $\hat{\gamma}_{i1} = \hat{\gamma}_{i2} = \ldots = \hat{\gamma}_{iN} = \bar{\gamma}_i$. If the error term u^t is non-zero, then the fact that u^t is distributed with zero mean does not necessarily imply

$$E\{\hat{\gamma}_{ij}\} = \gamma_i$$

If indeed $E\{\hat{\gamma}_{ij}\} \neq \gamma_i$ then clearly also $E\{\hat{\gamma}_i\} \neq \gamma$ and $\hat{\gamma}_{ij}$ and $\hat{\gamma}_i$ are known as biased estimators.

Computational aspects. The standard deviation of the additive error term u^t is specified as a percentage of the current population. Thus u^t is distributed normally as

$$u^t = N(0,\sigma_u^{(2)}) = N(0,\theta^{(2)}w^{t(2)}) \quad \ldots\ldots\ldots \quad [\ 3.12\ ]$$

where θ= standard deviation of the error term divided by the current population. It is clear that the error term is heteroscedastic.
The generating equation utilized to obtain the interregional population record W is thus

$$w^t = \Gamma w^{t-1} + v^t\ \theta\ w^t \quad \ldots\ldots\ldots\ldots \quad [\ 3.13\ ]$$

where v^t is a vector of random normal deviates (RND) distributed as N(0,1). Details of the computations for which results are presented in this Chapter are identified by the following set of symbols ("system identification");

N = sample size

n = length of the intercensal population record, in years

R_s = ratio of base populations

σ_u = standard deviation of the error term, specified as as a percentage of the current population(=100 θ)

A flow chart of the computer program is shown on Figure 5.

Autoregressive bias. It can be shown theoretically that the least squares estimates of Γ in an autoregressive model are consistent (and hence asymptotically unbiased).[2] Thus as $n \to \infty$ so will $E\{\hat{\gamma}_{ij}\} \to \gamma_i$ Hence $E\{\hat{\bar{\gamma}}_i\} \to \gamma_i$, independent of sample size.[3] For small n (length of the intercensal record) it is therefore to be anticipated that $E\{\hat{\bar{\Gamma}}\} \neq \Gamma$. Figure 6 shows selected values of $\hat{\bar{\gamma}}_{ij}$ plotted against n, and the bias for n=15 is indeed much smaller than for n=8. However, the asssumption of time-invariant migration rates is unrealistic over longer time periods, and for five and ten year series for which this assumption may be more realistic, the estimates of Γ are subject to significant auto-regressive error.

Weighted and unweighted estimation. Rogers (26) has shown that the heteroscedasticity implied by

$$u^t = N(0, \theta^{(2)} w^{t(2)}) \quad \text{[3.12]}$$

may be eliminated by dividing the estimating equation [3.8] by w_i

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[2] see e.g.Goldberger (27), p.273

[3] since Γ consists of the k arrays γ_i, we may also write $E\{\hat{\bar{\Gamma}}\} \to \Gamma$

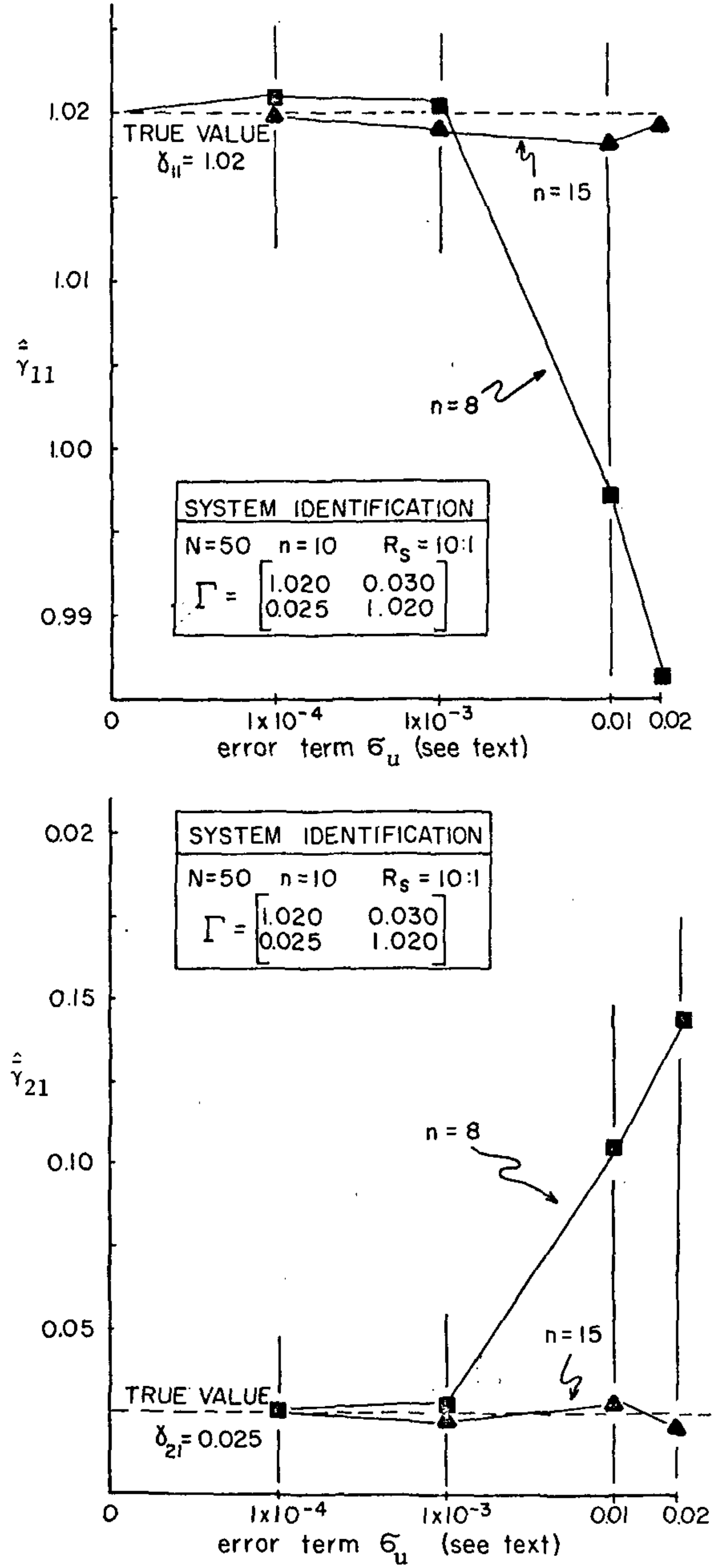

Figure 6 : Autoregressive bias as a function of record length and the magnitude of the additive error term.

$$(y_i \ominus w_i) = (W \ominus w_i)\ \gamma_i + (u^t \ominus w_i) \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 3.14\ ]$$

or $\quad y_i' = W'\ \gamma_i + u^{t'}$

for which $\quad E\{u'\ u'^T\} = \theta^{(2)} I$

Rogers stated, on the basis of a single example, that

> "...Weighted ULS estimates do not appreciably differ from the unweighted estimates. Thus it appears that weighting does not significantly alter the estimation results." (Rogers, op.cit.,p.527)

The evidence of numerous Monte Carlo runs would, surprisingly, tend to support this contention. Weighted estimation is recommended, but the difference is small. The results for a typical run are given on Table 1. To obtain an overall measure of the deviation of the estimated coefficients from the true values, Hotellings T^2 (elaborated in Appendix A) has been computed. Figure 7 shows that the T^2-statistics for weighted estimates lie below those for unweighted estimates. Thus whatever the significance level chosen to determine the critical region, i.e. for which $T^2 > T^2_{cr}$ upon which to test a null hypothesis of no bias, weighted estimates are more likely to lie within the non-critical region. Results also show certain of the off-diagonal elements of Γ to be markedly more sensitive to autoregressive bias than the diagonal elements. The estimates of γ_{21} of Table 1, for example, are all significantly biased (rejection of the null hypothesis of no bias using the univariate t-test), even though T^2 is non-critical for the estimated operator matrix as a whole (at the same significance level).

System Identification		
$N = 50$	$\Gamma = \begin{bmatrix} 1.00560 & 0.00300 \\ 0.05570 & 1.01925 \end{bmatrix}$	$n = 10$ $\quad R_s = 10 : 1$
$\sigma_u = 0.001\%$	$\begin{bmatrix} 1.00505 & 0.00305 \\ 0.07296 & 1.01781 \end{bmatrix}$	$\begin{bmatrix} 1.00480 & 0.00306 \\ 0.03612 & 1.02090 \end{bmatrix}$
$\sigma_u = 0.005\%$	$\begin{bmatrix} 1.00102 & 0.00343 \\ 0.13370 & 1.01284 \end{bmatrix}$	$\begin{bmatrix} 1.00394 & 0.00317 \\ 0.15216 & 1.01111 \end{bmatrix}$
$\sigma_u = 0.01\%$	$\begin{bmatrix} 0.98840 & 0.00456 \\ 0.31902 & 0.99709 \end{bmatrix}$	$\begin{bmatrix} 0.99440 & 0.00404 \\ 0.39701 & 0.99061 \end{bmatrix}$
	Least Squares $\hat{\hat{\Gamma}}$	MAD $\hat{\hat{\Gamma}}$

Table 1 Estimated interregional growth operators

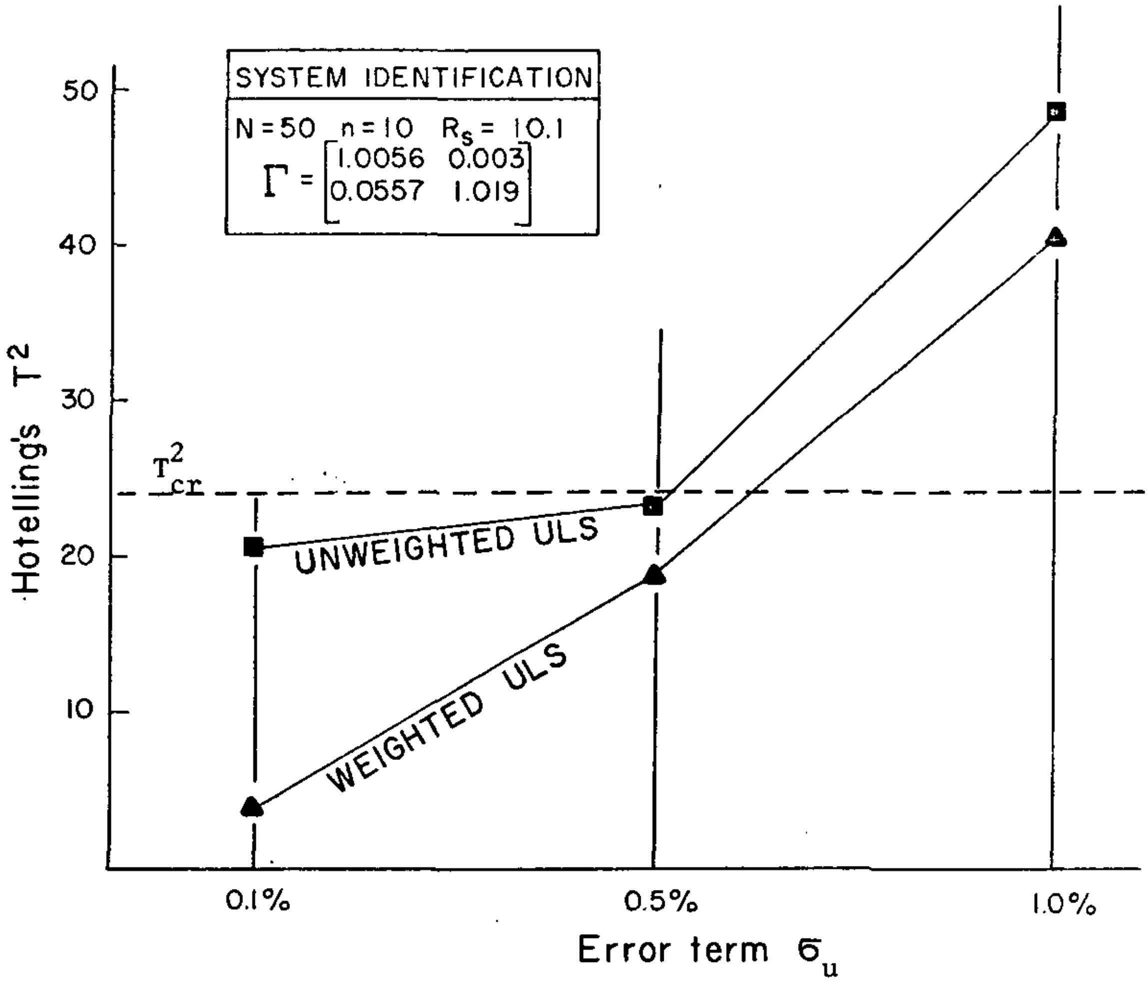

Figure 7 : Hotellings T^2 for weighted and unweighted ULS estimates.

MAD v. ULS Estimation. Little information is available on the statistical properties of MAD estimators. The papers by Karst (28) and Ashar and Wallace (29) studied simple schemes[4], and some loss of efficiency appeared significant (as compared to ULS estimates). The t-test was used to test the hypothesis of no bias for individual coefficient estimates, but none could be rejected.

Again using Hotellings T^2 statistic as an overall measure of deviation from the true values, it was found that MAD estimates were considerably more sensitive to autoregressive error than ULS, and that the weighting procedure was demonstrably deleterious. Figure 8 illustrates this contention graphically. However, no significant difference between MAD and LS was apparent in the variance of the estimates.[5] Table 2 shows a typical set of sample standard deviations of $\hat{\Gamma}$ for various modes of estimation.

Apart from the greater sensitivity to autoregressive error, certain computational disadvantages rule against the use of MAD as an estimation mode. In some samples of 50 coefficient estimates from synthetic data sets, up to 50 percent of the estimates needed to be discarded. In some cases the specified limit on the number of iterations in the simplex algorithm was attained, in others the solution basis contained error terms rather than growth operator elements (see Appendix B).

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[4]The systems $y=a+bx+u$ and $y=a+bX_1+cX_2+u$ respectively.
[5]As measured by the test for the equality of sample variance-covariance matrices elaborated in Appendix A.

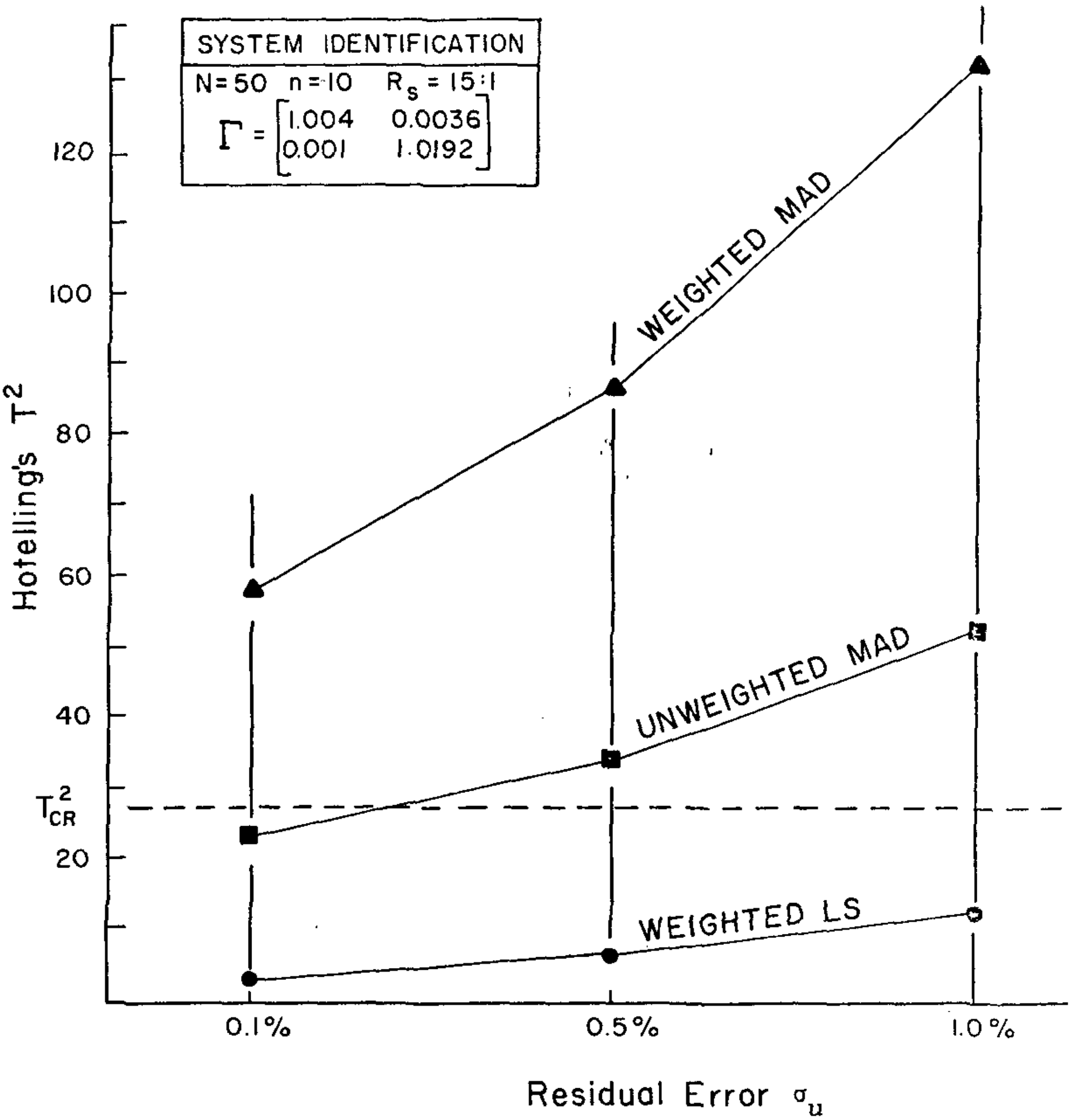


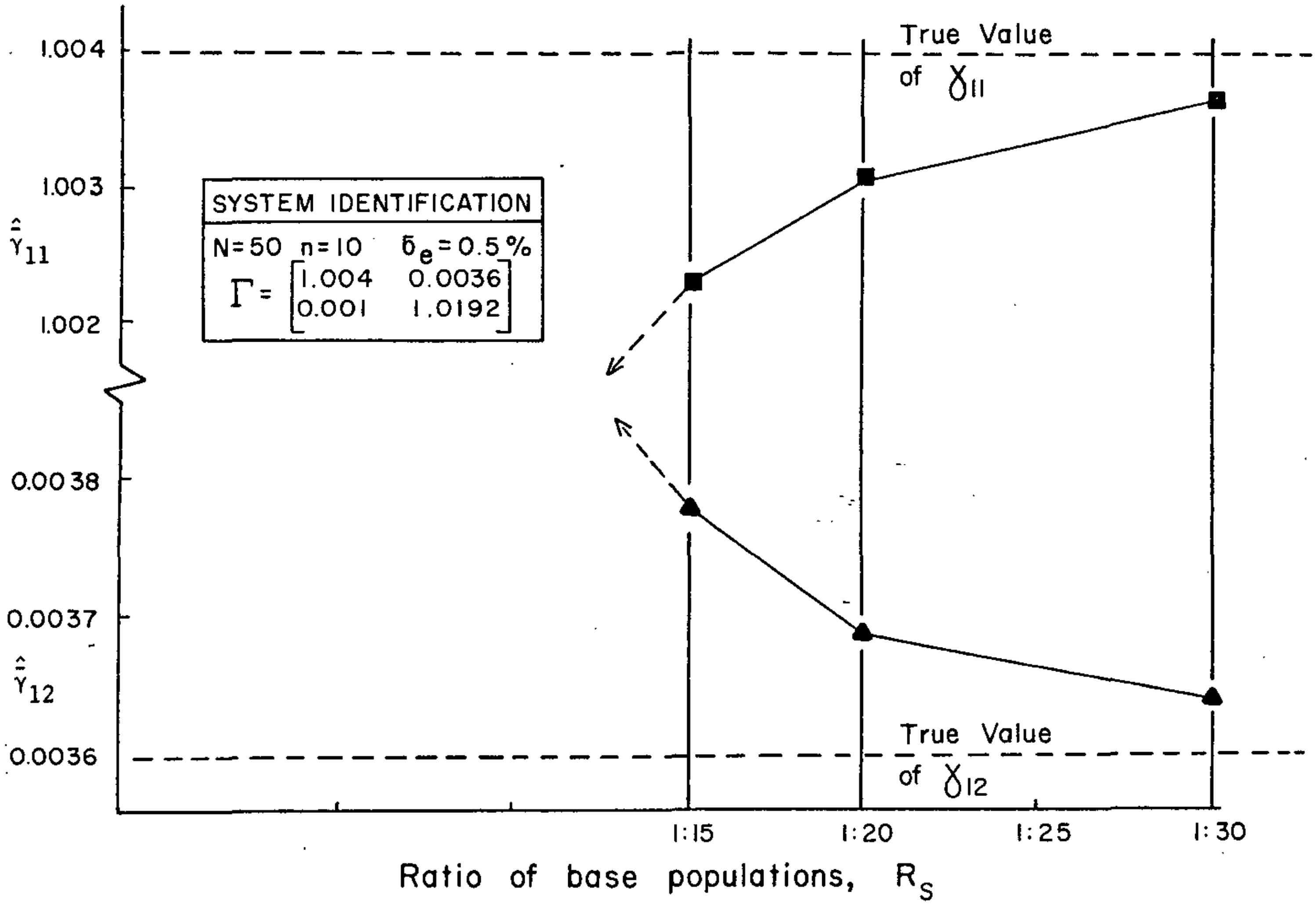
Figure 8 : Hotellings T^2 for MAD and ULS estimates

		[ ] = Var { $\hat{\Gamma}$ } x 10^{-3}	
	Error term	Weighted	Unweighted
LS	$\sigma_u = 0.5\%$	$\begin{bmatrix} 22.0 & 1.7 \\ 337.0 & 26.0 \end{bmatrix}$	$\begin{bmatrix} 23.6 & 1.8 \\ 284.5 & 22.3 \end{bmatrix}$
	$\sigma_u = 1.0\%$	$\begin{bmatrix} 47.0 & 3.6 \\ 513.0 & 39.0 \end{bmatrix}$	$\begin{bmatrix} 44.5 & 3.4 \\ 554.0 & 43.9 \end{bmatrix}$
MAD	$\sigma_u = 0.5\%$	$\begin{bmatrix} 30.0 & 2.3 \\ 187.0 & 14.8 \end{bmatrix}$	$\begin{bmatrix} 22.7 & 1.7 \\ 219.0 & 16.7 \end{bmatrix}$
	$\sigma_u = 1.0\%$	$\begin{bmatrix} 53.0 & 4.3 \\ 369.0 & 29.6 \end{bmatrix}$	$\begin{bmatrix} 39.0 & 3.1 \\ 419.0 & 33.0 \end{bmatrix}$

Table 2 Standard deviation of estimated growth operators for various estimation modes.

Dependence on relative population size. Figure 9 illustrates the dependence of the growth operator estimates for the smaller region in a two region set on the size of the larger region in the system. It is to be noted that under constant error specification, the estimated coefficients $\hat{\Gamma} \to \Gamma$ as the ratio of base populations, denoted R_s, increases. Ceteribus paribus, we anticipate that the estimates for the migration rates in the system Springfield-United States would be more accurate that the corresponding estimates for the system Springfield-Massachusetts.

Conclusions. The results of the Monte Carlo study show the Rogers method of estimating migration rates from an interregional population record to be unsatisfactory. An explanation of the poor results will be attempted in the following section. In addition to serious problems of bias, the variability of the estimators is unacceptable. Table 3 serves to emphasize this point once more. The standard deviations of the off-diagonal elements of $\hat{\Gamma}$ (i.e. the place-specific migration rates) are many orders of magnitude greater than the corresponding error introduced in generating the data. Examination of further results listed in Appendix C shows this to be quite general, and not limited to any one estimation mode.



Figue 9 : Dependence of growth operator estimates on the ratio of base populations

System Identification

$N = 50 \quad n = 10 \quad R_s = 1 : 10 : 20 \quad \sigma_u = 0.1\ \% \quad$ Weighted LS

$$\underset{\text{True growth operator}}{\Gamma} = \begin{bmatrix} 1.020 & 0.005 & 0.005 \\ 0.012 & 1.065 & 0.025 \\ 0.030 & 0.030 & 1.100 \end{bmatrix}$$

$$\underset{\text{Estimated growth operator}}{\hat{\Gamma}} \pm s_{(\hat{\Gamma})} = \begin{bmatrix} 1.014 \pm 0.02 & 0.007 \pm 0.006 & 0.004 \pm 0.001 \\ 0.059 \pm 0.08 & 1.047 \pm 0.03 & 0.034 \pm 0.014 \\ 0.022 \pm 0.05 & 0.033 \pm 0.02 & 1.099 \pm 0.005 \end{bmatrix}$$

Table 3 Unrestricted least squares estimate of a three-region system growth operator.

Analysis of the Rogers Model

Variance of the least squares estimator $\hat{\gamma}$. The variance of the least squares estimator is given by

$$\mathrm{Var}\{\hat{\gamma}\} = E\{(\hat{\gamma} - E\{\hat{\gamma}\})(\hat{\gamma} - E\{\hat{\gamma}\})^T\}. \quad \ldots\ldots\ldots \quad [\ 3.15\ ]$$

On the assumption that $E\{\hat{\gamma}\} = \gamma$, we may write

$$\mathrm{Var}\{\hat{\gamma}\} = E\{(\hat{\gamma} - \gamma)(\hat{\gamma} - \gamma)^T\} \quad \ldots\ldots\ldots \quad [\ 3.16\ ]$$

Now

$$\begin{aligned} \gamma - \hat{\gamma} &= \gamma - (W^TW)^{-1} W^T y \\ &= \gamma - (W^TW)^{-1} W^T(W\gamma + u) \\ &= \gamma - (W^TW)^{-1} W^T W\gamma - (W^TW)^{-1} W^T u \\ &= -(W^TW)^{-1} W^T u \end{aligned}$$

hence

$$\mathrm{Var}\{\hat{\gamma}\} = (W^TW)^{-1} W^T E\{u\ u^T\} W (W^TW)^{-1} \quad \ldots\ldots \quad [\ 3.17\ ]$$

Suppose further that W is in the weighted form (see Eq.[3.12]) for which

$$E\{u\ u^T\} = \theta^2 I$$

hence

$$\mathrm{Var}\{\hat{\gamma}\} = \theta^2 (W^TW)^{-1} \quad \ldots\ldots\ldots\ldots\ldots \quad [\ 3.18\ ]$$

The magnitude of the elements of the variance-covariance matrix is thus dependent on $(W^TW)^{-1}$. This may be written as

$$(W^TW)^{-1} = \frac{\mathrm{Adjoint}(W^TW)}{|W^TW|}$$

from which follows that if $|W^TW|$ is small, $\mathrm{Var}\{\hat{\gamma}\}$ will be large. It is therefore of interest to examine the conditions for which $|W^TW|$ is indeed small. Consider a two-region system for which W^TW may be written out as

$$W^TW = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} [w_1 \quad w_2] = \begin{bmatrix} w_1^T w_1 & w_1^T w_2 \\ w_1^T w_2 & w_2^T w_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sum w_1^{t(2)} & \sum w_1^t w_2^t \\ \sum w_1^t w_2^t & \sum w_2^{t(2)} \end{bmatrix} \quad \ldots\ldots\ldots\ldots [\ 3.19\ ]$$

Suppose now that one population record (i.e. one column of W) can be written in terms of the *other, for* example as

$$w_1^t = b\ w_2^t + u^t \quad \ldots\ldots\ldots\ldots [\ 3.20\ ]$$

where u^t is a random term such that $E\{u^t\} = 0$. Thus

$$W^TW = \begin{bmatrix} \sum(bw_2^t + u^t)^{(2)} & \sum(bw_2 + u^t)\ w_2^t \\ \sum(bw_2^t + u^t)\ w_2^t & \sum w_2^{t(2)} \end{bmatrix} \quad \ldots\ldots [\ 3.21\ ]$$

for which it is easily shown that

$$|W^TW| = \sum u^{t(2)} \sum w_2^{t(2)} \quad \ldots\ldots\ldots\ldots [\ 3.21\ ]$$

which is independent of b. Thus if u is small (implying a strong relation between the two columns of W), then $|W^TW|$ is also small and the resultant variance is large. Although we have seen that the estimates are biased, i.e. $E\{\hat{\gamma}\} \neq \gamma$, and thus Eq.[3.16] does not hold strictly, we infer that the large variance of the growth operator estimates is due at least in part to strong multicolinearity in the intercensal population record W.

Perturbation Analysis. As an alternative approach, consider the effect of a given change in the data matrix, say dW on the resultant change in the solution vector, say d . Of particular interest is the relative change in $\hat{\Gamma}$ expressed in terms of the relative change in the data matrix W. By utilizing norms we may express the ratio of the relative changes as a scalar, namely

$$f = \frac{\dfrac{||d\hat{\Gamma}||}{||\hat{\Gamma}||}}{\dfrac{||dW||}{||W||}} \quad \ldots\ldots\ldots\ldots\ldots \quad [\ 3.22\ ]$$

where f is defined as the amplification factor, and where the 1-norm as defined by Fadeeva (32) is given by

$$||A||_1 = \max_j \sum_i |a_{ij}| \qquad ||x||_1 = \sum |x_i|$$

Table 4 shows the results of some actual numerical computations for two-region systems. The computed values of Cond(W^TW), the condition number, lie between 27000 and 28000, indicative of the ill-condition of the W^TW matrix.[4] The systems are described in column 1 (of table 4), and the 1-norm of the perturbation dW indicated in column 2. Column 3 shows the resulting estimate of the interregional growth operator, with the change in Γ (as compared to the unperturbed system) shown in column 4. Column 5 shows the 1-norm of dΓ and column 6 the amplification factor f. The magnitude of the amplification factors encountered is indicative of the sensitivity of the system to small errors.

---

[4] see for example Forsythe and Moler (31) or Albasiny (30)

1	dy 2	$\hat{\Gamma}$ 3	$d\hat{\Gamma}\ x10^{-5}$ 4	$\|\|d\hat{\Gamma}\|\| x10^{-5}$ 5	f 6
Unperturbed System		1.01380 0.00228 -0.02640 1.01687			
Perturbed System A (=1955 pop. from 13004 to 13049)	50	1.01236 0.00240 -0.02513 1.01676	144 12 124 105	268	86
Perturbed System B (=1951 to 1960 pops. incremented by +5	50	1.01194 0.00244 -0.02609 1.01684	187 16 35 76	222	71
Perturbed System C (=1951 to 1960 pops. incremented alternately by ±5)	50	1.01387 0.00227 -0.02582 1.01682	7 1 62 5	69	22
Perturbed System D (=1955 pop. incremented by +100)	100	1.01016 0.00259 -0.02683 1.01691	364 311 398 4	762	122

Table 4 Perturbation analysis, least squares estimates of the Rogers Model (see text for explanation of tabulations)

Errors in the Data

As a consequence of the ill condition of the W^TW matrix small changes in the data will result in large changes in the estimated coefficient matrix. Suppose that the true interregional population record is given by W^*, but due to errors in measurement (enumeration error) a data set $W \neq W^*$ is utilized for estimation purposes. This problem is common to econometrics, where considerable effort has been devoted to its resolution. Three approaches have been suggested, namely

1. The classical approach, in which analytical expressions are derived on the basis of restrictive assumptions about the probability distributions of the errors involved. Results show, in general, an underestimation of the true coefficients. The mathematics for a multivariate problem, even in the absence of autoregression, are quite intractable, and were not further explored.

2. Instrumental variables, widely used in econometrics, which do yield consistent estimates. The use of vital statistics as an instrumental variable will be examined in Chapter V

3. Grouping methods, based on the grouping of observations and less restrictive assumptions about the error terms. Of these we shall investigate the method proposed by Wald (35).

---

[5]Johnston (33), for example, devotes an entire chapter to the errors in variables problem.

<u>Wald's method</u>[6] Suppose we wish to estimate β from some linear model given by

$$y_i = \alpha + \beta x_i + u_i \quad \text{[3.23]}$$

$$E\{u_i\} = 0$$

when both y_i and x_i are measured with error. If the observation pairs are arranged such that the x_i stand in ascending order

$$x_1 \leq x_2 \leq . \quad . \quad . \leq x_n$$

and the observations are divided into two equal groups containing the smaller and larger x-values respectively, each group of m=n/2 pairs (x_i, y_i), then Wald's estimator of β is given by

$$\hat{\beta}_W = \frac{\bar{y}_2 - \bar{y}_1}{\bar{x}_2 - \bar{x}_1} \quad \text{[3.24]}$$

where $\bar{x}_i$ and $\bar{y}_i$ define the center of gravity of the groups given by

$$\bar{x}_1 = \frac{1}{m}\sum^{m} x_i \qquad \bar{y}_1 = \frac{1}{m}\sum^{m} y_i$$

$$\bar{x}_2 = \frac{1}{m}\sum^{m} x_{i+m} \qquad \bar{y}_2 = \frac{1}{m}\sum^{m} y_{i+m}$$

Thus $\hat{\beta}_W$ is the slope of the line connecting the two centres of gravity. Wald's estimator of α is given by

$$\hat{\alpha}_W = \bar{y} - \hat{\beta}_W \bar{x} \quad \text{[3.25]}$$

---

[6] The brief recapitulation of Wald's method follows Theil and Yzeren (34). For the original exposition see Wald (35)

where $\bar{x}$, $\bar{y}$ defines the centre of gravity of the entire point set (x_i, y_i). Wald's method has the merit of computational simplicity over least squares, and, more relevant in our context, yields consistent estimates of β. Its disadvantage lies in a loss of efficiency, which is dependent on the distribution of the x_i. For the special case of a rectangular distribution, appropriate for a uniformly spaced time series, Bartlett (36) has shown that, given

$$E\{u_i u_j\} = 0 \ , i \neq j$$

$$E\{u_i^2\} = \sigma$$

then $$\operatorname{Var}\{\hat{\beta}_W\} = \sigma^2 \frac{1/m_1 + 1/m_3}{(\bar{x}_3 - \bar{x}_1)^{(2)}} \quad \ldots \ldots \ldots \ldots [\ 3.26\ ]$$

is minimized if both the left hand and right hand groups contain one-third of the data pairs, the central third not utilized for the estimation of β_W.

Wald's method has been extended to two independent variables by Hooper and Theil (37). Just as two centres of gravity are sufficient to determine a straight line in the x-y plane, so will three points determine a plane in 3-space.

Application of Wald's method to the Rogers Model. For a k-region model, it follows that the data need be subdivided into (k+1) groups. However, the (k+1)-dimensional solution surface (a plane for k=2) is restricted to pass through the origin since the Rogers Model demands zero intercept. Thus we may use LS on the (k+1) centres of gravity subject to the prior hypothesis $\alpha = 0$.

Figure 11 demonstrates this method. To effect an objective comparison with least squares, the perturbed systems of table 4 were re-estimated by the above modification to Wald's method, and the results tabulated on Table 5. As expected, the error amplification is less dependent on the distribution of errors than by LS estimation. However the order of magnitude of the amplifcations remains similar, and the estimated coefficients themselves have not changed significantly. In particular, negative coefficients are still present. Although but few systems were evaluated by this method, the overall similarity of results to least squares did not justify the programming effort of a full Monte Carlo study.

---

[7] Experiments with different grouping arrangements, e.g. (m_1=2, m_2=6, m_3=2) in place of (m_1=3, m_2=4, m_3=3) showed that the most evenly partioned groupings gave best results.

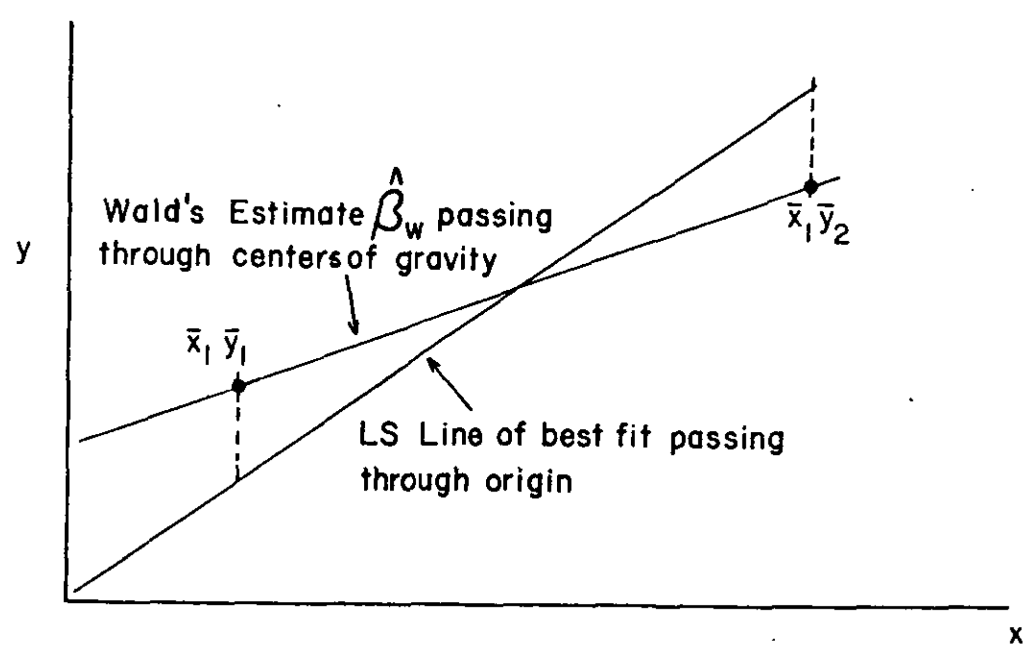


Figure 11 : Wald's method under the prior hypothesis of zero intercept.

System	Amplification Factor f	
	Wald's Method	Least Squares (see table 4 column 6)
A	57	86
B	52.5	71
C	58	22
D	66	122

Table 5 Comparison of Wald's Method and least squares estimates of the Rogers Model interregional growth operators.

CHAPTER IV

MIGRATION RATES AS RANDOM VARIABLES

Specification Error

In actual computations for sets of regions (say a central city and a subset of suburbs), an additive error of the type analysed in Chapter III is incurred by virtue of migration into the system from excluded regions. Let the population record for the k included and m excluded regions be partitioned as

$$\underset{(n \times k+m)}{W^*} = [\underset{(n \times k)}{W}, \underset{(n \times m)}{\tilde{W}}]$$

and let the interregional growth operator for all (m+k) regions, denoted Γ^*, be partitioned as

$$\Gamma^* = \begin{bmatrix} \underset{(k \times k)}{\Gamma} & \underset{(k \times m)}{\tilde{\Gamma}_1} \\ \underset{(m \times k)}{\tilde{\Gamma}_3} & \underset{(m \times m)}{\tilde{\Gamma}_2} \end{bmatrix} \quad \ldots\ldots\ldots\ldots [\ 4.1\ ]$$

Let us suppose that the error term u represents no longer enumeration error as in the previous chapter, but represents the error introduced by virtue of migrations to and from the m omitted regions. Thus in place of

$$y_i = W\gamma_i + u \quad \ldots\ldots\ldots\ldots\ldots\ldots [\ 4.2\ ]$$

we have $$y_i = W\gamma_i + \tilde{W}\tilde{\gamma}_{1i} \quad \ldots\ldots\ldots\ldots\ldots [\ 4.3\ ]$$

where $\tilde{\gamma}_{1i}$ is the transpose of the i-th row of $\tilde{\Gamma}_1$. Now the least squares estimate of γ_i as obtained from Eq.[4.2] is given by

$$\hat{\gamma}_i = (W^T W)^{-1} W^T y_i \quad . . . . [4.4]$$

To compare this estimate with obtained from Eq.[4.2], premultiply Eq.[4.2] by W^T to obtain

$$W^T W \gamma_i = W^T y_i - W^T \tilde{W} \tilde{\gamma}_{1i} \quad . . . . [4.5]$$

hence $$\gamma_i = \hat{\gamma}_i - (W^T W)^{-1} W^T \tilde{W} \tilde{\gamma}_{1i} \quad . . . . [4.6]$$

from which it is evident that $\hat{\gamma}_i = \gamma_i$ if 1) $\tilde{\gamma}_{1i} = 0$, implying no migratory movements to the omitted regions (i.e. model is specified correctly) or 2) $W^T\tilde{W} = 0$. The Rogers Model demands that the least squares hyperplane pass through the origin, and thus the computations use variables in original rather than deviation form. Hence $W^T\tilde{W}$ will always be positive. The restriction of zero intercept will be relaxed in Chapter V, and then using deviation form the condition $W^T\tilde{W} = 0$ would imply that the population record of each and every included region be "uncorrelated" with the population record of each and every excluded region. In view of the multicolinearity noted in Chapter III, this is improbable.

Let us replace Eq.[4.3] by the more realistic specification

$$y_i = W \gamma_i + \tilde{W} \tilde{\gamma}_{1i} + u \quad . . . . [4.7]$$

for which enumeration error is included in addition to the error incurred by omission of the m regions. With W^* partitioned as [W , $\tilde{W}$] then the least squares estimate of the i-th row of Γ^* ($i \leq k$) is given by

$$\begin{bmatrix} \hat{\gamma}_i \\ \hat{\gamma}_{1i} \end{bmatrix} = \begin{bmatrix} W^TW & W^T\tilde{W} \\ W^T\tilde{W} & \tilde{W}^T\tilde{W} \end{bmatrix}^{-1} \begin{bmatrix} W^Ty \\ \tilde{W}^Ty \end{bmatrix} \quad \text{. . . . . [4.8]}$$

Applying the partitioned inversion rule it can be shown that[1]

$$\begin{bmatrix} \hat{\gamma}'_i \\ \hat{\gamma}'_{1i} \end{bmatrix} = \begin{bmatrix} (W^TW)^{-1}W^Ty - (W^TW)^{-1}W^T\tilde{W} D^{-1} \tilde{W}^TC\, y \\ D^{-1}\tilde{W}^TC\, y \end{bmatrix} \quad \text{. . . . . . [4.9]}$$

where $C = I - W\,(W^TW)^{-1}\,W^T$

$D = \tilde{W}^T\, C\, \tilde{W}$

and where the superscript ' indicates that the estimates $\hat{\gamma}'_i, \hat{\gamma}'_{1i}$ are derived from the correctly specified equation [4.7]. It follows directly that

$$\hat{\gamma}'_i = (W^TW)^{-1}\, W^T\, y_i - (W^TW)^{-1}W^T\, \tilde{W}\, \hat{\gamma}'_{1i}$$

$$= \hat{\gamma}_i - (W^TW)^{-1}W^T\, \tilde{W}\, \hat{\gamma}'_{1i}$$

or $$\hat{\gamma}_i = \hat{\gamma}'_i + (W^TW)^{-1}W^T\, \tilde{W}\, \hat{\gamma}'_{1i} \quad \text{. . . . . [4.10]}$$

and thus the least squares estimate from the incorrectly specified equation [4.2] equals the least squares estimate from the correctly specified equation [4.7] again only if $\hat{\gamma}_{1i} = 0$ or if $W^T\tilde{W} = 0$.

From the above elaborations it is evident that specification error may be eliminated by including in an interregional system an

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[1]Goldberger (27),p.27 and p.174, who obtains this result in connection with stepwise regression procedures.

additional region that represents the sum of all hitherto excluded regions. In the study of migration movements in the Springfield area, inclusion of a system "rest of the world" would thus eliminate specification error. In practice, for reasons of comparability of data, the region "rest of the world" must be replaced by the region "rest of the United States", reducing specification error to overseas immigration and emmigration.

Migration Rates as Random Variables

Autoregressive formulation. The assumption of constant migration rates, as required by least squares estimation, is not met in reality. Assuming therefore time dependence of migration rates, let μ^t_{ij} be the elements of M^t at time t. A demographically plausible and mathematically not intractable assumption is that successive annual migration rates μ^t_{ij}, μ^{t-1}_{ij}, are drawn from a bivariate normal population with means $\bar{\mu}^t_{ij} = \bar{\mu}^{t-1}_{ij} = \bar{\mu}_{ij}$ and standard deviations $\sigma^t_{ij} = \sigma^{t-1}_{ij} = \sigma_{ij}$ and with serial correlation coefficient ρ_{ij}. It is hypothesized that migration rates fluctuate about some mean value over an n-year period, with deviations from the mean exhibiting strong serial correlation. A higher than average annual rate will thus most probably be succeeded by another higher than average rate, depending on the magnitude of the serial correlation coefficient. The conditional expectation of such a process may be given as

$$E\{\mu^t_{ij} \mid \mu^{t-1}_{ij}\} = \bar{\mu}_{ij} + \rho_{ij}(\mu^{t-1}_{ij} - \bar{\mu}_{ij}) \qquad [\ 4.11\ ]$$

and

$$\mathrm{Var}\{\mu^t_{ij} \mid \mu^{t-1}_{ij}\} = \sigma^2_{ij}\ (1 - \rho^2_{ij}) \qquad [\ 4.12\ ]$$

Fiering (38) has shown that the sequence

$$\mu^t_{ij} = \bar{\mu}_{ij} + \rho_{ij}\ (\mu^{t-1}_{ij} - \bar{\mu}_{ij}) + v^t\ \sigma_{ij}\ (1 - \rho^2_{ij})^{0.5} \qquad [\ 4.13\ ]$$

has the properties [4.7] and [4.8] and where v^t is a random normal deviate distributed as N(0,1). Recalling that $\Gamma = M$ for $i \neq j$, and and assuming that a similar serial dependence exists for the diagonal elements of Γ, we may write in matrix notation for a k-regional model

$$\Gamma^t = \bar{\Gamma} + R \circledast (\Gamma^{t-1} - \bar{\Gamma}) + V^t \circledast \Sigma \circledast R^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 4.14\ ]$$

where Γ^t = interregional growth operator at time t

$\bar{\Gamma}$ = mean growth operator over the n-year interval

R = matrix of serial correlation coefficients ρ_{ij}

V^t = matrix of random normal deviates v_{ij}^t at time t

Σ = matrix of standard deviations σ_{ij}

R^* = matrix of elements $(1 - \rho_{ij}^2)^{0.5}$

Monte Carlo Study. To investigate this time dependence in its effect on the least squares estimation process, a second Monte Carlo simulation was executed. Data was generated by sequential application of the relation

$$w^t = \Gamma^t w^{t-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 4.15\ ]$$

where Γ^t is given by Eq.[4.14]. The resultant least squares estimates $\hat{\bar{\Gamma}}$ were then compared to the mean growth operator $\bar{\Gamma}$ utilized in generating the data as in the Monte Carlo study of Chapter III.

Figure 12 shows the dependence of $\hat{\bar{\gamma}}_{ij}$ as a function of $\sigma(\gamma_{ij})$, the standard deviation of a particular growth operator element and the serial correlation coefficient. We note that weak serial correlation reverses the direction of bias, but stronger serial correlation again reverts to the direction of bias observed for zero serial correlation. At some optimum value of the serial correlation coefficient, the estimators are unbiased. This is consistent with the effects of auto-

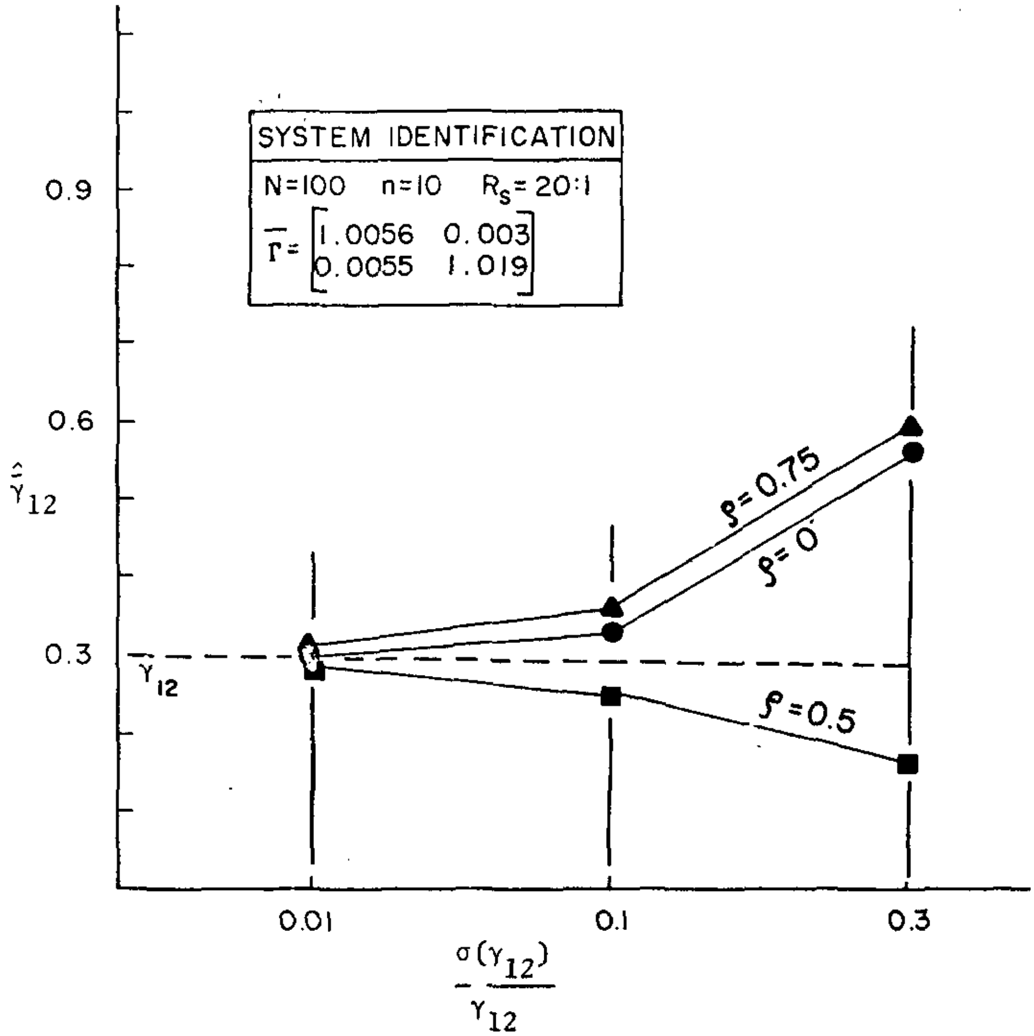

Figure 12 : Estimated growth operator elements as a function of growth operator variability and serial correlation

correlated additive errors in an autoregressive equation. But since the degree of serial correlation is unknown *a priori*, this phenomenon cannot be applied to adjust the estimated $\hat{\gamma}$ in an actual computation.

Application of the T^2 and t tests again proved inconclusive with respect to detecting significant differences between weighted and unweighted estimation. However, $Var\{\hat{\Gamma}\}$ was significantly lower for the weighted estimates.

Conclusions. In the Monte Carlo studies of Chapters III and IV we have examined the effects of violating the requisite assumptions for unbiased least squares estimation of the interregional growth operators. These violations are known to exist in the light of present knowledge of migration behaviour and enumeration accuracy. The information obtained from these Monte Carlo Simulations will be utilized in subsequent Chapters in the design of a computational model that recognizes the limitations of demographical data more fully than does the original formulation of Rogers, and that recognizes the stochastic nature of the growth operator elements.

CHAPTER V

AN ITERATIVE ESTIMATION MODEL

Introduction

As we have seen in the two previous Chapters, demographic reality violates the assumptions of the statistical estimation model, in particular that of an error-free data record. The results of Chapter IV, where migration rates were considered as serially correlated random variables, suggest that the least squares technique will yield unbiased estimates of the mean growth operator over the estimation interval if the data record could be appropriately smoothed. Rogers (26) did experiment with a simple smoothing scheme, but was unable to answer the question of what constituted the "best" degree of smoothing:

> "...This (sensitivity) underscores the importance of establishing a more rational method for smoothing the data points than is presented here."(Rogers, op. cit.,p.529)

In this Chapter an iterative estimation model will be developed for which the optimum degree of smoothing is well-defined.

First we turn, however, to a consideration of how certain relations between the elements of the growth operator may be utilized to obtain more accurate estimates of Γ. The assumption of homogenous propensity will also be relaxed.

A Restricted Least Squares Estimator

Derivation. To illustrate the development of the restricted estimation model, consider the equations governing a two-region population system

$$w_1^{t+1} = \gamma_{11}\, w_1^t + \mu_{21}\, w_2^t$$

$$w_2^{t+1} = \mu_{12}\, w_1^t + \gamma_{22}\, w_2^t \quad \ldots\ldots\ldots\ldots \quad [\ 5.1\ ]$$

Decomposing the γ_{ii} into their constituent parts (see Eq.[3.6]),

$$w_1^{t+1} = (1+\beta_1-\delta_1-\omega_1)\, w_1^t + \mu_{21}\, w_2^t$$

$$w_2^{t+1} = \mu_{12}\, w_1^t + (1+\beta_2-\delta_2-\omega_2)\, w_2^t \quad \ldots\ldots \quad [\ 5.2\ ]$$

and recalling that $\beta_1\, w_1^t = b_1^t$ and $\delta_1\, w_1^t = d_1^t$ (Eq.[3.1] and [3.2])

$$w_1^{t+1} - b_1^t + d_1^t = (1-\omega_1)\, w_1^t + \mu_{21}\, w_2^t$$

$$w_2^{t+1} - b_2^t + d_2^t = \mu_{12}\, w_1^t + (1-\omega_2)\, w_2^t \quad \ldots\ldots \quad [\ 5.3\ ]$$

For this two-region system, the net outmigration rate ω_1 clearly equals the place specific rate μ_{12} (since there is only one destination for outmigrants). For the general k-region case

$$\sum_{\substack{j=1 \\ j\neq i}}^{k} \mu_{ij} = \omega_i \quad \ldots\ldots\ldots\ldots \quad [\ 5.4\ ]$$

i.e. the net outmigration rate equals the sum of the place specific rates for any one column of M. In order to recognize this restriction on the coefficients of the growth operator it is necessary to estimate all k rows of Γ simultaneously. Also, since births and deaths are available from published vital statistics records and can be subtracted from the left hand side of the estimating equation (i.e. Eq.[5.3] for a two-region system) prior to any numerical computations, we shall redefine y_i such that

$$\underset{(n \times 1)}{y_i} = \begin{bmatrix} w_i^{t+1} - b_i^{t+1} + d_i^{t+1} \\ w_i^{t+2} - b_i^{t+2} + d_i^{t+2} \\ \cdot \\ \cdot \\ w_i^{t+n} - b_i^{t+n} + d_i^{t+n} \end{bmatrix}$$

The simultaneous estimation of all k rows of Γ requires the estimating equation to be rewritten as

$$\begin{bmatrix} y_1 \\ \hline y_2 \\ \hline \\ \hline y_k \end{bmatrix} = \left[\begin{array}{c|c|c|c} W & 0 & & 0 \\ \hline 0 & W & & 0 \\ \hline & & & \\ \hline 0 & 0 & & W \end{array}\right] \begin{bmatrix} \gamma_1 \\ \hline \gamma_2 \\ \hline \\ \hline \gamma_k \end{bmatrix} \quad . \; . \; . \; . \; . \; [\ 5.5\ ]$$

which can be written in a more compact notation as

$$\underset{(kn \times 1)}{y_K} = \underset{(kn \times k^2)}{W_K} \; \underset{(k^2 \times 1)}{\gamma_K} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 5.6\ ]$$

The difference between the capital K subscript, which indicates a simultaneous estimation of Γ, and the lower case k subscript, which indicates the number of regions in a system, should be fully noted.

Now for the two-region case considered above, the restriction [5.4] can be written in matrix form as

$$\underset{R}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \underset{\gamma_K}{\begin{bmatrix} 1-\omega_1 \\ \mu_{21} \\ \mu_{12} \\ 1-\omega_2 \end{bmatrix}} = \underset{e_2}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \quad \ldots\ldots\ldots \; [\ 5.7\ ]$$

This prior restriction can be incorporated into the least squares estimation procedure as follows. For the general k-region case, we seek a γ_K, say $\hat{\gamma}_K^*$, that minimizes

$$s = (y_K - W_K\gamma_K)^T (y_K - W_K\gamma_K) \quad \ldots\ldots\ldots \; [\ 5.8\ ]$$

subject to $R\ \gamma_K = e_k$

where R is a matrix that is partitioned as

$$R = [\ I_{k_1} \quad I_{k_2} \quad . \; . \; . \quad I_{k_k}\ ]$$

This problem may be formulated as the minimization of

$$s = (y_K - W_K\gamma_K)^T (y_K - W_K\gamma_K) - 2\lambda'(R\ \gamma_K - e_k) \quad \ldots \; [\ 5.9\ ]$$

where λ' is the appropriate (k x 1) vector of Lagrange multipliers.[1]

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[1]The Lagrange multiplier approach to prior information regression was first suggested by Dwyer(39). See also Theil (40) and Chipman and Rao (41).

Thus $$s = y_K^T y_K - 2\gamma_K W_K^T y_K + \gamma_K^T W_K^T W_K \gamma_K - 2\lambda'(R\gamma_K - e_k) \quad [5.10]$$

hence $$\frac{\partial s}{\partial \gamma_K} = -2 W_K^T y_K + 2 W_K^T W_K \gamma_K - 2 R^T \lambda' \quad . \; . \; . \; . \; . \; . \quad [5.11]$$

$$\hat{\gamma}_K^* = (W_K^T W_K)^{-1} W_K^T y_K + (W_K^T W_K)^{-1} R^T \lambda^* \quad . \; . \; . \; . \; . \quad [5.12]$$

It can be shown that[2]

$$\hat{\gamma}_K^* = (W_K^T W_K)^{-1} W_K^T y_K + (W_K^T W_K)^{-1} R^T [R (W_K^T W_K)^{-1} R^T]^{-1} \cdot$$
$$(e_k - R (W_K^T W_K)^{-1} W_K^T y_K) \quad . \; . \; . \; . \quad [5.13]$$

$\hat{\gamma}_K^*$ will be referred to as the restricted estimator. It can also be shown that there has been a gain in efficiency over the unrestricted estimator $\hat{\gamma}_K$, since $\mathrm{Var}\{\hat{\gamma}_K^*\} < \mathrm{Var}\{\hat{\gamma}_K\}$.[3]

Relaxation of the homogenous migration propensity assumption. Implicit in the Rogers Model formulation of interregional migration is the concept of homogenous migration propensity, in that the migration rates μ_{ij} are assumed to operate on the entire population. Prior knowledge of migration behaviour, however, indicates that this assumption is erroneous since certain age and skill groups are very much more susceptible to migration than others. Let us assume that the population consists of two segments, one potentially mobile, the other not

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[2]Goldberger (27), p.257 and Judge and Takayama (42)
[3]Goldberger (27), p.258 or Theil (40)

susceptible to migration during the interval considered. Let this latter segment be denoted z_i. Then we may rewrite Eq.[5.3] as

$$w_1^{t+1} - b_1^t + d_1^t = (1 - \omega_1)(w_1^t - z_1) + \mu_{21}(w_2^t - z_2) + z_1$$

$$w_2^{t+1} - b_2^t + d_2^t = \mu_{12}(w_1^t - z_1) + (1 - \omega_2)(w_2^t - z_2) + z_2$$

[5.14]

but since the z_i are assumed constant over the estimation interval

$$w_1^{t+1} - b_1^t + d_1^t = (z_1 - (1-\omega_1) z_1 - \mu_{12} z_2) + (1-\omega_1) w_1^t + \mu_{12} w_2^t$$

$$w_2^{t+1} - b_2^t + d_2^t = (z_2 - (1-\omega_2) z_2 - \mu_{12} z_1) + \mu_{12} w_1^t + (1-\omega_2) w_2^t$$

which is identical to Eq.[5.3] with the addition of the constant terms

$$\alpha_1 = \omega_1 z_1 - \mu_{21} z_2$$

$$\alpha_2 = \omega_2 z_2 - \mu_{12} z_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [\ 5.15\ ]$$

But for the two-region case, $\omega_1 = \mu_{12}$ and $\omega_2 = \mu_{21}$. Hence $\alpha_1 = -\alpha_2$. For the three-region case we may write

$$\alpha_1 = \omega_1 z_1 - \mu_{21} z_2 - \mu_{31} z_3$$

$$\alpha_2 = \omega_2 z_2 - \mu_{12} z_1 - \mu_{32} z_3$$

$$\alpha_3 = \omega_3 z_3 - \mu_{13} z_1 - \mu_{23} z_2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [\ 5.16\ ]$$

hence $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and generalizing to the k-region case we have by induction

$$\sum_{i=1}^{k} \alpha_i = 0$$

This additional restriction on the set of estimated coefficients will be added to the restriction set [5.4]. In matrix notation, Eq.[5.14] becomes

$$w^{t+1} - b^t + d^t = \alpha + \Gamma w^t \qquad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ [\ 5.17\ ]$$

and hence the corresponding estimating equation for the i-th row of Γ is

$$\underset{(n \times 1)}{y_i} = \underset{(n \times k+1)}{[\ e_n \quad W\ ]} \underset{(k+1 \times 1)}{\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}} \qquad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ [\ 5.18\ ]$$

To simplify notation, we shall still write

$$y_K = W_K \gamma_K$$

for the simultaneous estimation of equation [5.18] for all k rows of Γ, but noting that W has been augmented by the n-unit vector and that the first element of $\hat{\gamma}_i$, now of dimensions (k+1 x 1), represents the intercept term $\hat{\alpha}_i$. Computationally there is no difference between Eq. [5.18] and Eq.[3.7]. The new restriction set for a two-region system is now

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 1 - \omega_1 \\ \mu_{21} \\ \alpha_2 \\ \mu_{12} \\ 1 - \omega_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad . \ . \ . \ . \ . \ [\ 5.19\ ]$$

and for the k-region case we have

$$\underset{(k+1 \times k(k+1))}{\underset{R}{[\, \underset{1}{I_{k+1}} \quad \underset{2}{I_{k+1}} \quad . \; . \; . \quad \underset{k}{I_{k+1}} \,]}} \underset{(k(k+1) \times 1)}{\underset{\gamma_K}{\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \\ \gamma_k \end{bmatrix}}} = \underset{(k+1 \times 1)}{\underset{r}{\begin{bmatrix} 0 \\ e_k \end{bmatrix}}} \quad . \; . \; . \quad [\ 5.20\ ]$$

The restricted estimator $\hat{\gamma}_K^*$ thus contains the estimated intercept terms $\hat{\alpha}_i$. However, it is also necessary to obtain the z_i before we can compute the magnitude of the migration streams. For example, the number of migrants from region j to region i during the estimation interval, m_{ji}, is given by

$$m_{ji} = \sum_{t=1}^{n} \mu_{ji} \ (w_i^t - z_i) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [\ 5.21\ ]$$

from which it is clear that the z_i must be known explicitly. Unfortunately the system of equations represented by [5.16] for the three-region case is singular, and cannot be solved uniquely for z_i in terms of the α_i. Suppose, however, that that segment of the population not susceptible to migration can be approximated by some constant ratio of the base population. For example, let $z_i = c\, w_i^o$ where c is some positive constant. Then it follows from Eq.[5.15], for the two-region case, that

$$\alpha_1 = \omega_1\, c\, w_1^o - \mu_{21}\, c\, w_2^o \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [\ 5.22\ ]$$

and $$c = \frac{\hat{\alpha}_1}{\hat{\omega}_1 w_1^o - \hat{\mu}_{21} w_2^o}$$

hence $$\hat{z}_1 = \frac{\hat{\alpha}_1 w_1^o}{\hat{\omega}_1 w_1^o - \hat{\mu}_{21} w_2^o}$$

which is easily generalised to the k-region case, namely

$$\hat{z}_i = \frac{\hat{\alpha}_i w_i^o}{\hat{\omega}_i w_i^o - \sum_{\substack{j=1 \\ j \neq i}}^{k} \hat{\mu}'_{ji} w_i^o} \qquad \text{[5.23]}$$

Iterative Improvement

<u>Intercensal Population estimates</u>. The restricted estimation technique elaborated above requires an intercensal population record that is consistent with the assumptions of the interregional model. Thus a simple geometric or linear interpolation for each of the regions in the interregional set will clearly result in zero estimates for the off-diagonal elements of the growth operator. Unadjusted post-censal estimates may contain considerable discontinuities, and the use of the midyear population estimates published in the Massachusetts Vital Statistics Documents proved quite unfruitful.

A more productive approach lies in the modification of the modern techniques of <u>post</u>-censal population estimation for the purpose of retrospective <u>inter</u>censal estimation by elimination of the closure error. Examination of such methods quickly showed the regression methods to possess the prerequisite consistency with the interregional formulation. Amongst the more recent contributions to such techniques, Zitter and Shryock (43) demonstrated the power of regression methods in a comparative study conducted by the US Census Bureau. Rosenberg (44) reported that the ratio-correlation technique was the most reliable available at the county level, and Pursell (45) showed that the latter's predictive power could be enhanced by augmenting the conventional symptomatic indicators of population (births, deaths, automobile registrations, non-agricultural employment) with dummy variables and stratification by economic base characteristics.

The correlation methods generally depend on a multivariate regression of the form

$$w_i^n = \sum_{j=1}^{p} a_j X_{ij} \quad . . . . [\ 5.24\ ]$$

where w_i^n is the population of the i-th region for the census year n, X_{ij},j=1, . . p is a set of symptomatic variables and a_j the corresponding set of regression coefficients.

Consider, for example, the use of vital statistics (births, deaths) as symptomatic indicators. We postulate that there is a linear relation between increases in births and deaths and an increase in population. Perfect proportionality would imply

$$\frac{w_i^t - w_i^1}{w_i^n - w_i^1} = \frac{a_1}{a_1+a_2} \frac{b_i^t - b_i^1}{b_i^n - b_i^1} + \frac{a_2}{a_1+a_2} \frac{d_i^t - d_i^1}{d_i^n - d_i^1} \quad . . . [\ 5.25\ ]$$

from which

$$w_i^t = w_i^1 + \frac{w_i^n - w_i^1}{a_1+a_2} \left\{ a_1 \frac{b_i^t - b_i^1}{b_i^n - b_i^1} + a_2 \frac{d_i^t - d_i^1}{d_i^n - d_i^1} \right\} \quad . . . [\ 5.26\ ]$$

where w_i^t = population of the i-th region at time t

w_i^1 = population of the i-th region at time 1 (=census population at the beginning of the n-year interval)

w_i^n = population of the i-th region at time n (=census population at the end of the n-year interval)

b_i^t ,d_i^t = allocated births, deaths for region i during the t-th time period

b_i^n, b_i^1, d_i^n, d_i^1 = allocated biths and deaths for region i in the census years n, 1 respectively.

and where the weighting coefficients a_1, a_2 are derived from the multiple regression of population growth over the total interval on the corresponding increase of births and deaths, namely

$$(w_i^n - w_i^1) = a_1 \, (b_i^n - b_i^1) + a_2 \, (d_i^n - d_i^1) \quad i=1,2,\ldots,k \quad [\ 5.27\ ]$$

Births and deaths must also be regarded as random variables, and the variations in annual vital rates may be considerable on a local area basis. These variations are considered in more detail in Chapter VI. A more representative value for a census year is thus given, for example, by application of a 5-point moving average to the series of vital statistics about that census year, namely

$$\bar{b}_i^n = \frac{1}{5}\sum_{\ell=n-2}^{n+2} b_i^\ell \qquad \bar{d}_i^n = \frac{1}{5}\sum_{\ell=n-2}^{n+2} d_i^\ell \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad [\ 5.28\ ]$$

and analogously for $\bar{b}_i^1$ and $\bar{d}_i^1$. The regression equation [5.27] uses these smoothed values to estimate the weighting coefficients a_j.

It follows from Eq.[5.26] that changes in the crude birth and death rates during the intercensal interval do not affect the resultant population estimates provided such changes are linear over the entire interval. Changes in the crude rates result from changes in mortality (for which the above assumption is not unreasonable), changes in fertility (for which the assumption is less reasonable), or from changes in the age structure resulting from age selective migration (for which the assumption is more questionable). Nevertheless, numerical results justify the intuitive notion that the estimation of crude, average migration operators is insensitive to non-linear changes in vital rates.

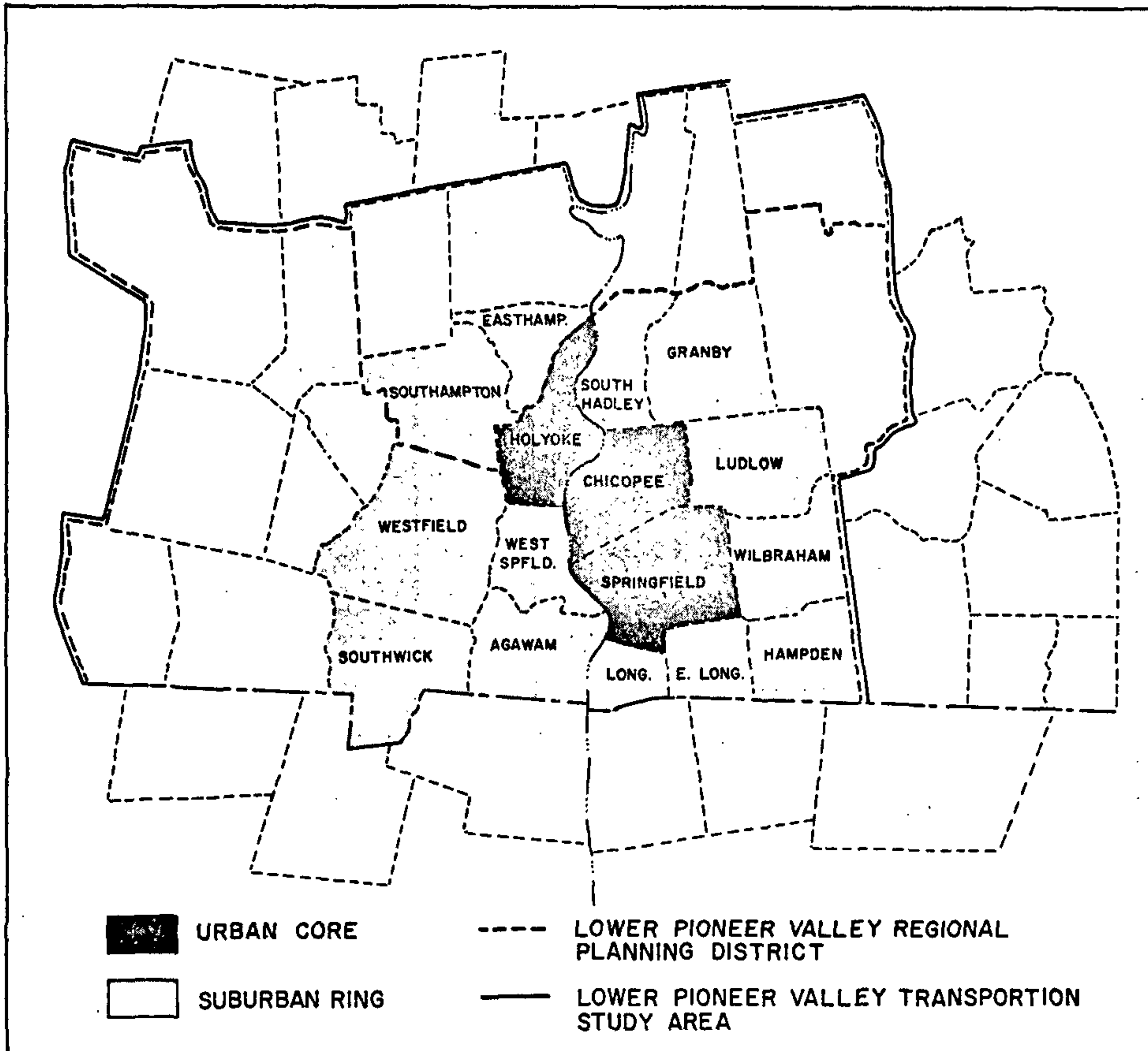

Figure 13 : Map of the Lower Pioneer Valley Regional Planning District

Iterative Improvement Algorithm I. The assumption of constant vital and migration rates (or linearly variant) implies that for each and every column of W the sequence of values

$$w_i^t,\ w_i^{t+1} \ldots,\ w_i^{t+n}$$

be either monotonically increasing or monotonically decreasing. That is, the differences

$$(w_i^{t+1} - w_i^t),\ t = 1, 2, \ldots, n$$

must all be positive (for an increasing population), all negative (for a decreasing population) or all zero (in the case of a completely stagnant population).[3] Examination of Figure 14 shows this requirement to be violated for the typical local area, even using the 5-point moving average. Consequently, the restricted estimator $\hat{\gamma}_K^*$ will still yield poor results, reflected in estimated coefficients of improbable magnitude. Some measure of adjusting this initial intercensal estimate is demanded such that the aformentioned conditions are fulfilled. A smoothing scheme would be desirable, since this would concurrently eliminate the deleterious effects of an erroneous data record, and smooth the fluctuations in the data resulting from the stochastic nature of the growth operator elements.

Introducing the notation

$$\pi(W_K, R) = (W_K^T W_K)^{-1} R^T [R\ (W_K^T W_K)^{-1} R^T]^{-1} \quad \ldots\ldots [\ 5.29\ ]$$

---

[3]These conditions hold also, of course, to the ordered sequences of births and deaths.

and recalling that

$$\hat{\gamma}_K = (W_K^T W_K)^{-1} W_K^T y_K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 5.30\ ]$$

then Eq.[5.13] may be written in the abbreviated form

$$\hat{\gamma}_K^* = \hat{\gamma}_K + \pi(W_K, R)(e - R\hat{\gamma}_K) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 5.31\ ]$$

Suppose now that the true growth operator, Γ, were known. Then application of Γ to the known base population would yield an intercensal population estimate W^*. The least squares estimate of γ_K, denoted as usual by $\hat{\gamma}_K^*$, is then given by

$$\hat{\gamma}_K^* = (W_K^{*T} W_K^*)^{-1} W_K^{*T} y_K^* + \pi(W_K^*, R)(e - R(W_K^{*T} W_K^*)^{-1} W_K^{*T} y_K^*)$$

$$[\ 5.32\ ]$$

But since this ideal data set is error free

$$(W_K^{*T} W_K^*)^{-1} W_K^{*T} y_K^* = \gamma_K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 5.33\ ]$$

and hence $$\hat{\gamma}_K^* = \gamma_K + \pi(W_K^*, R)(e - R\gamma_K) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 5.34\ ]$$

Also, the restriction set [5.20] is exact, i.e.

$$R\gamma_K = e$$

and thus $$\hat{\gamma}_K^* = \gamma_K + \pi(W_K^*, R)\ 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 5.35\ ]$$

$$= \gamma_K$$

This simple result suggests the following iterative improvement scheme. Given some initial estimate of the intercensal population record, $W_{(0)}$, evaluate $$\hat{\gamma}^*_{K(0)} = \hat{\gamma}_{K(0)} + \pi(W_{K(0)},R)(e - R\hat{\gamma}_{K(0)}) \quad . \quad . \quad . \quad . \quad . \quad [\ 5.36\ ]$$

and let Λ be the sum of the absolute values of the array of corrections, namely

$$\Lambda(\hat{\gamma}^*_{K(0)}) = \sum|\pi(W_{K(0)},R)(e - R\hat{\gamma}_{K(0)})| \quad . \quad . \quad . \quad . \quad . \quad [\ 5.37\ ]$$

By some transformation, denoted $g(W_{K(0)})$, obtain a second estimate $W_{K(1)}$ and recompute

$$\hat{\gamma}^*_{K(1)} = \hat{\gamma}_{K(1)} + \pi(W_{K(1)},R)(e - R\hat{\gamma}_{K(1)}) \quad . \quad . \quad . \quad . \quad . \quad [\ 5.38\ ]$$

$$\Lambda(\hat{\gamma}^*_{K(1)}) = \sum|\pi(W_{K(1)},R)(e - R\hat{\gamma}_{K(1)})| . \quad . \quad . \quad . \quad . \quad . \quad [\ 5.39\ ]$$

$$W_{K(1)} = g(W_{K(0)})$$

Now from Eq.[5.35] it follows that

$$\Lambda(\gamma_K) = 0$$

and thus a logical criterion for the best intercensal population estimate, say $W_{K(Opt)}$, is such that

$$\Lambda(\gamma^*_{K(i)}) = \text{minimum} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 5.40\ ]$$

Such a criterion presupposes that the successive transformation given by $g(W_{K(i)})$ does in fact yield a series of values of Λ that possess a minimum. If such a transformation can be found, then the iterative computation commenced in Eq. [5.36] - [5.39] is continued until a minimum value of Λ is attained. If the first minimum so encountered is the global minimum, then $\hat{\gamma}^*_{K(Opt)}$ is such that

$$\Lambda(\hat{\gamma}^*_{K(0)}) > \Lambda(\hat{\gamma}^*_{K(1)}) > \cdot \quad \cdot \quad \cdot > \Lambda(\hat{\gamma}^*_{K(opt)}) \leq \Lambda(\hat{\gamma}^*_{K(opt+1)})$$

The intercensal population record W is composed of k columns, one per region. The transformation $W_{(i)} = g(W_{(i-1)})$ is therefore k-dimensional, namely

$$(w_{1(1)}, w_{2(1)}, \cdot \cdot \cdot\ w_{k(1)}) = g(w_{1(0)}, w_{2(o)}, \cdot \cdot \cdot\ w_{k(0)})$$

The problem may be visualized geometrically as a multidimensional response-surface optimization in k-dimensional transformation space. The literature on response surface optimization is quite extensive, and of the proven techniques available to locate the optimum (i.e. the minimum value of Λ that corresponds to a particular number of transformations), the discrete-step, steepest descent procedure appears appropriate. Although this technique requires that Λ be a convex function over the transformation space, for which we are assured that any located minimum represents the desired global optimum, there are significant computational advantages over algorithms requiring less restrictive conditions. In particular, fewer iterations are generally required than by uniform grid (complete search) or random sampling methods.[5]

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[5]For an exhaustive consideration of the theoretical and computational considerations involved, see Wilde (46) and Hill and Hunter (47). The only application of this technique in the environmental engineering literature appears to be Hufschmidt (48) in the context of a response surface representing net benefits to a multiple-use reservoir system, the axes representing the different inputs (units of flood control storage, units of irrigation capacity etc.)

Experiments with a number of data transformations showed a three-point moving average transformation on the columns of W to yield a sequence of values of Λ that did indeed possess a minimum and for which the corresponding estimate of the growth operator was consistent with prior knowledge of migration behaviour.

Consider, for example, a two-region system for which an initial estimate of the interregional population record is given by

$$W_{(0,0)} = (w_{1(0)}, w_{2(0)})$$

where the bracketed subscript indicates the number of smoothing operations on each column of W. Then evaluate

$$\Lambda(\hat{\gamma}^*_{K(0,0)}), \quad \hat{\gamma}^*_{K(0,0)} = (W^T_{K(0,0)} W_{K(0,0)})^{-1} W^T_{K(0,0)} y_{K(0,0)}$$

$$\Lambda(\hat{\gamma}^*_{K(0,1)}), \quad \hat{\gamma}^*_{K(0,1)} = (W^T_{K(0,1)} W_{K(0,1)})^{-1} W^T_{K(0,1)} y_{K(0,1)}$$

$$\Lambda(\hat{\gamma}^*_{K(1,0)}), \quad \hat{\gamma}^*_{K(1,0)} = (W^T_{K(1,0)} W_{K(1,0)})^{-1} W^T_{K(1,0)} y_{K(1,0)}$$

where $W_{K(0,0)}$ = block diagonal matrix of blocks $W_{(0,0)}$

$$W_{(0,1)} = (w_{1(0)}, w_{2(1)})$$

$$W_{(1,0)} = (w_{1(1)}, w_{2(0)})$$

In words, $\Lambda(\hat{\gamma}^*_{K(1,0)})$ is that value of the objective function obtained by smoothing once the intercensal population record of region 1, region 2 unsmoothed.

Then evaluate

$$\mathrm{Max}\{\Lambda(\hat{\gamma}^*_{K(0,1)}) - \Lambda(\hat{\gamma}^*_{K(0,0)}) \, , \, \Lambda(\hat{\gamma}^*_{K(1,0)}) - \Lambda(\hat{\gamma}^*_{K(0,0)})\}$$

to obtain the next starting point. Thus, for example, if

$$\Lambda(\hat{\gamma}^*_{K(1,0)}) - \Lambda(\hat{\gamma}^*_{K(0,0)}) > \Lambda(\hat{\gamma}^*_{K(0,1)}) - \Lambda(\hat{\gamma}^*_{K(0,0)})$$

the next starting point is given by the coordinates 1,0 in the transformation space (corresponding to $W_{(1,0)}$), and the next iteration computes the estimates

$$\Lambda(\hat{\gamma}^*_{K(2,0)}), \quad \hat{\gamma}^*_{K(2,0)} = (W^T_{K(2,0)} W_{K(2,0)})^{-1} \, W^T_{K(2,0)} \, y_{K(2,0)}$$

$$\Lambda(\hat{\gamma}^*_{K(1,1)}), \quad \hat{\gamma}^*_{K(1,1)} = (W^T_{K(1,1)} W_{K(1,1)})^{-1} \, W^T_{K(1,1)} \, y_{K(1,1)}$$

and again selects the maximum

$$\mathrm{Max}\{\Lambda(\hat{\gamma}^*_{K(2,0)}) - \Lambda(\hat{\gamma}^*_{K(1,0)}) \, , \, \Lambda(\hat{\gamma}^*_{K(1,1)}) - \Lambda(\hat{\gamma}^*_{K(1,0)})\}$$

to obtain another starting point. The iterations are continued in this manner until a minimum is attained. Figure 14 shows such a path in two-space as manifested by the computer algorithm for the two-region set Massachusetts- West Springfield. The optimum estimate of γ_K is given by the intercensal estimate $W_{(4,15)}$.

Figure 15 shows the initial population estimate for West Springfield, $w_{2(0)}$, which is based on proportionality to the smoothed record of vital statistics over the intercensal interval, and the final intercensal estimate $w_{2(15)}$ for which the optimum was obtained.

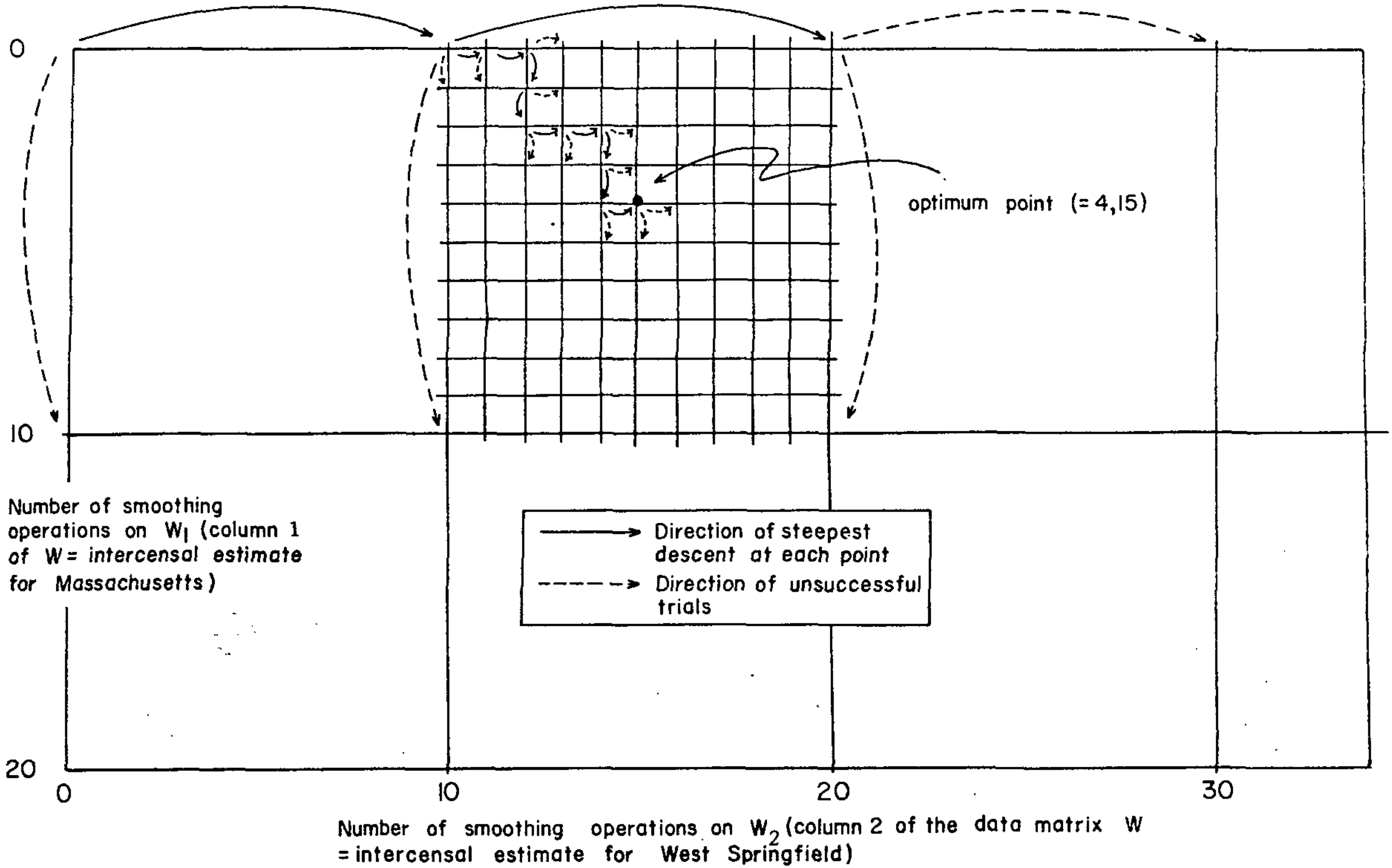

Figure 14 : Discrete-step, steepest descent algorithm for the two-region system Massachusetts-W.Springfield.

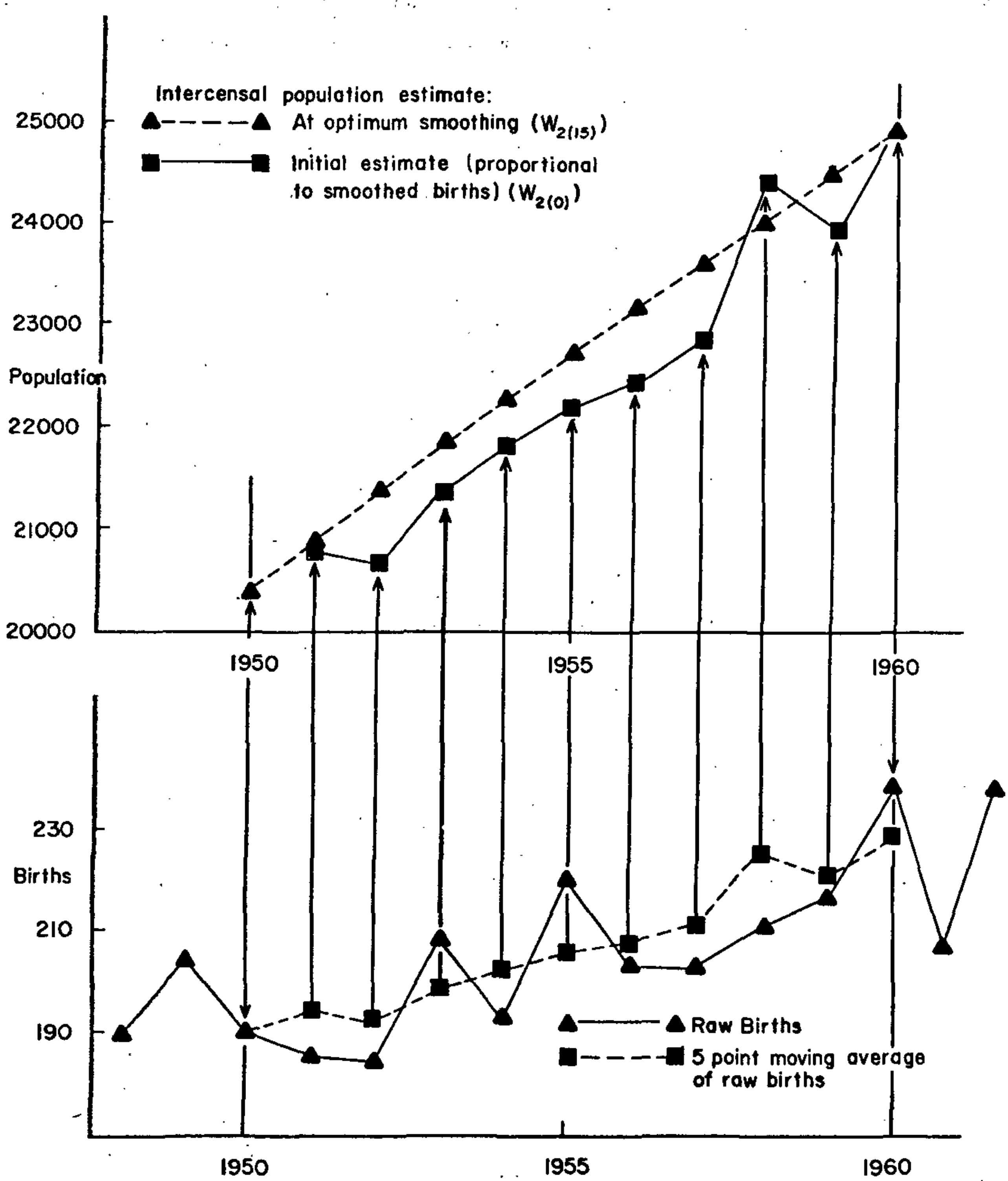

Figure 15 : Initial and optimum estimates of intercensal population for West Springfield, 1950-1960

Iterative Improvement Algorithm II. An alternative objective function may be formulated by comparison of the net migration vector generated by the estimated growth operator to an independent estimate of net migration obtained by the so-called vital statistics method.[6] The (k x 1) vector of net migrations, denoted ℓ, is given by this latter method as

$$\ell = w^n - w^o - \sum_{t=1}^{n} b^t + \sum_{t=1}^{n} d^t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 5.41\ ]$$

which is readily computed from the allocated birth and death statistics available for local areas. On the basis of an estimated growth operator, the resultant net migration vector is given by the expression

$$\hat{\ell} = \hat{\Gamma}^{(n)} w^o + \sum_{i=1}^{n} \hat{\Gamma}^{(n-i)} (b^i - d^i) - w^o - \sum_{i=1}^{n} b^i + \sum_{i=1}^{n} d^i \qquad [\ 5.42\ ]$$

and therefore another criterion for the quality of the growth operator estimate is

$$\Theta(\hat{\gamma}_K^*) = |\hat{\ell} - \ell| = \text{minimum} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 5.43\ ]$$

Again using the steepest descent discrete-step algorithm to locate the minimum of Θ, the optimum estimate of γ_K will be obtained for which the deviation of the resultant net migration vector to the vital statistics method estimate is minimized.

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[6]See e.g. Siegal and Hamilton (49)

[7]This formulation is inexact, since allocated births and deaths are published on an annual basis(January 1-December 31), whereas the census population is for April.

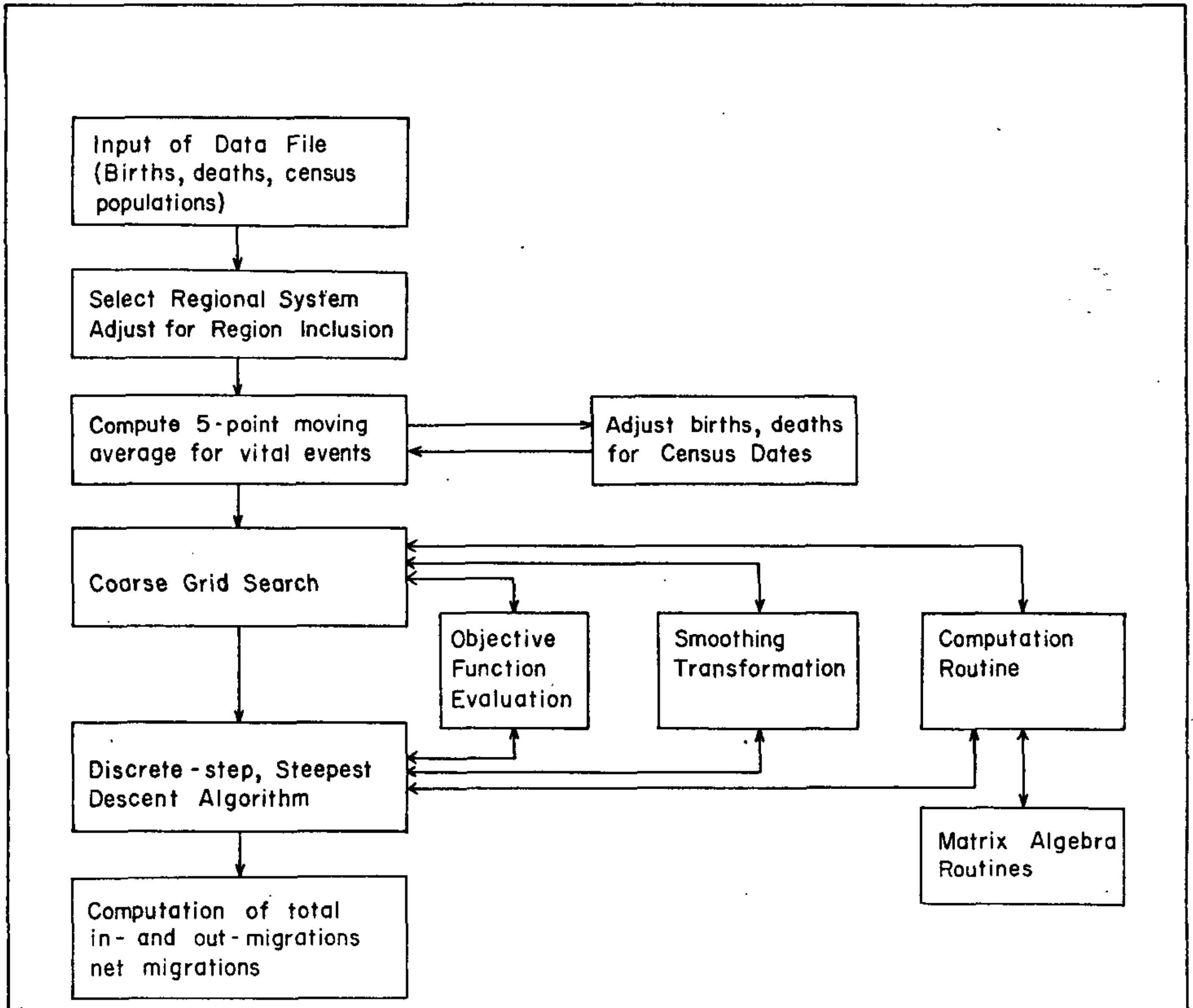

Figure 16 : Flow chart, Iterative estimation model (PROGRAM GRADP)

<u>Computational Aspects</u>. Preliminary studies showed the necessity of using double precision arithmetic due to the large differences in the order of magnitude of the regressors.[8] Sample coefficient matrices (estimated interregional growth operators) are given on Table 6 at the optimum point for both normal and double precision arithmetic. It is evident that the constraint set [5.20] is not satisfied in the former case. These discrepancies are reflected also in the subsequent estimates for total in- and outmigrants. For example, the normal precision estimate for total inmigrants during the period 1955-1960 for West Springfield is 3208, against 4023 for the double precision estimate. For 3-region systems, double precision was even more essential despite the significant increase in execution time.

Experiments showed further that the convergence of the discrete step, steepest descent algorithm was accelerated by fitting a polynomial to the five-point moving average that defines the initial intercensal estimate of W, prior to the first iteration. The least squares criterion differs from standard practice insofar as we require the polynomial to pass through the data points of the census years. Let the m-th degree polynomial be represented by

$$\zeta(t) = \sum_{\ell=0}^{m} a_\ell \, t^\ell \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [\ 5.44\ ]$$

Then the problem is to minimize

$$s = \sum^{t} [\sum_{\ell=0}^{m} a_\ell \, t^\ell \; - \; \tilde{w}_i^t]^{(2)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad [\ 5.45\ ]$$

subject to $\sum a_\ell = 1, \; a_o = 0$

---

[8] Double precision arithmetic is significant to 15 rather than 7 figures.

Normal Precision Estimate				Double Precision Estimate			
$\hat{\Gamma}$	0.99906	0.03840	-578.48	$\hat{\Gamma}$	0.99984	0.03000	-34.21
	0.00808	0.98653	438.20		0.00015	0.96999	34.21
Σ	1.00714	1.02493	140.28	Σ	0.99999	0.99999	0.000

Table 6 Normal and double precision arithmetic estimates of the growth operator for the two-region system Massachusetts-West Springfield.

	Estimated growth operator $\hat{\Gamma}$
No smoothing, origin of transformation space [0 , 0]	$\begin{bmatrix} 0.994144 & 0.676254 & 13139.0 \\ 0.005855 & 0.323745 & -13139.0 \end{bmatrix}$
Grid search minimum [0 , 75]	$\begin{bmatrix} 0.999774 & 0.037126 & 128.73 \\ 0.000225 & 0.962874 & -128.73 \end{bmatrix}$
Final optimum [0 , 84]	$\begin{bmatrix} 0.999840 & 0.037126 & -34.21 \\ 0.000159 & 0.969999 & 34.21 \end{bmatrix}$

Table 7 Estimated growth operators at the origin, grid search minimum and final optimum for the two-region system Massachusetts-West Springfield.

and where

$$\tilde{w}_i^t = \frac{w_i^t - w_i^1}{w_i^n - w_i^1}$$

The general solution for an m-th degree polynomial is given by Meier (50), to which paper the reader is referred for further details, since the solution is too cumbersome to be reproduced here.

A problem that created serious difficulty was the frequent occurence of negative coefficients in the estimated inter-regional growth operators of 3-region systems. These are, of course, unacceptable in the framework of interregional migration rates. Experiments with the MAD estimator showed that where least squares yielded a negative coefficient, the corresponding variable did not enter the optimal solution basis of the MAD simplex algorithm: by implication the corresponding migration rate is zero, which is generally improbable.

The iterative improvement algorithms were therefore modified such that the domain of transformation space that results in negative coefficients in the growth operator estimate be impassable. The origin no longer suffices as the starting point for the general case, and hence a coarse grid search determined that point on the lattice for which the objective function was found to be minimized, subject to the resultant coefficients being positive. The steepest descent algorithm commences at this point, but is no longer permitted to enter the domain of smoothing operations that results in negative coefficients, even if the objective function would thereby be lowered. Since the transformation space is discrete (the number of smoothing operations on any column of the data matrix is integer-valued), the optimum is still well-defined.

Results. Experiments with real data for the thirty communities of the Lower Pioneer Regional Planning District showed the proposed technique to give unsatisfactory results for the small rural towns, many of which contain fewer than 1,000 inhabitants. Elements of the estimated growth operators proved to be of improbably large magnitude, or to include negative coefficients over the entire smoothing domain. The three cities that comprise the core of the SMSA (Springfield, Holyoke, Chicopee) also yielded erratic results. However, for the thirteen towns about the central city that comprise the suburban ring (see Figure 13), the iterative, restricted least squares estimator gave uniformly good results.

The 1960 census data includes a statistic for the 1955 place of residence (of persons five years or older in 1960). Respondents residing in SMSA's indicated "same" or "different house" as their 1955 place of residence, and, if different, one of the following categories; "Central City of the SMSA" (Category I), "Other part of the SMSA (II), "outside the SMSA" (III) or "abroad" (IV). A respondent who changed his residence within the same town is thus classified together with those who entered from other parts of the SMSA (i.e. category II). The total number of in-migrants during the period 1955-1960 can thus only be given as a range. The lower bound assumes that all in category II changed residence within the town, the higher bound assumes that all in category II entered the town from other

	Net migration 1950-1960 Vital statistics 1	Net migration 1950-1960 ALG. I 2	Total inmigrants 1955-1960 As per 1960 Census 3	Total inmigrants 1955-1960 ALG. I 4	Total outmigrants 1955-1960 ALG. I 5
AGAWAM	3,574	3,841	3,406-4,492	4,275	2,527
EASTHAMPTON	164	391	*	5,466	5,316
EAST LONGMEADOW	4,242	4,710	2,678-3,533	4,069	1,880
GRANBY	1,689	1,905	*	5,776	5,039
HAMPDEN	786	883	859-995	1,191	622
LONGMEADOW	3,564	4,036	3,035-4,268	7,181	5,018
LUDLOW	3,402	3,682	2,456-4,601	5,272	3,648
SOUTH HADLEY	2,777	3,333	*	3,661	2,046
SOUTHAMPTON	588	215	*	355	251
SOUTHWICK	1,331	187	*	598	381
WESTFIELD	2,490	8,951	3,646-9,467	12,160	8,643
WEST SPRINGFIELD	1,253	1,464	4,020-9,081	3,836	3,181
WILBRAHAM	2,693		1,890-2,828		

Table 8 Results for two-region systems, Algorithm I

	Net migration 1950-1960	Net migration 1950-1960	Total inmigrants 1955-1960	Total inmigrants 1955-1960	Total outmigrants 1955-1960
	Vital statistics	ALG. II	As per 1960 Census	ALG. II	ALG. II
	1	2	3	4	5
AGAWAM	3,574	3,509	3,406-4,492	3,070	1,365
EASTHAMPTON	164	167	*	1,323	1,243
EAST LONGMEADOW	4,242	4,288	2,678-3,533	2,937	846
GRANBY	1,689	1,717	*	1,799	1,090
HAMPDEN	786	786	859-995	764	403
LONGMEADOW	3,564	3,689	3,035-4,268	2,443	600
LUDLOW	3,402	3,518	2,456-4,601	4,016	2,480
SOUTH HADLEY	2,777	2,794	*	2,406	1,035
SOUTHAMPTON	588	588	*	579	300
SOUTHWICK	1,331	1,350	*	1,147	504
WESTFIELD	2,490	1,255	3,646-9,467	4,017	3,486
WEST SPRINGFIELD	1,253	1,138	4,020-9,081	4,023	3,507
WILBRAHAM	2,693	2,786	1,890-2,828	1,904	533

Table 9 Results for two-region systems, Algorithm II

parts of the SMSA. This range does permit some evaluation of the numerically estimated results, and is given in column 3 of Tables 8 and 9.

Tables 8 and 9 show the results for two-region systems in double precision arithmetic. Agreement of columns 4 and 5 is excellent; it must be remembered that column 5 represents the 1955-1960 estimate as obtained from the average rate over the 10-year interval 1950-1960.

We note also that Algorithm I yields good estimates of net migration over the 10-year interval as compared to the vital statistics method, which contains some element of error (see Footnote 7). Since Algorithm II uses the vital statistics method in evaluating the objective function, it will inherently yield comparable net migration estimates at the optimum. It does appear that Algorithm I gives consistently higher estimates for in- and outmigrations than Algorithm II.

Results for three-region systems. By constraining the iterative algorithms to that smoothing domain for which the resultant estimated coefficients are positive, the principal computational difficulty was eliminated. Nevertheless, the results are somewhat unsatisfactory. Table 10 shows the estimated ratios of inmigrants from Hampden County (to a given town) to inmigrants from areas outside Hampden County. These ratios are obtained from the third row of the estimated growth operator for the three-region systems Massachusetts-Hampden County-Suburban town. The results are erratic,

	Estimated ratio of inmigrants from other parts of Hampden County to inmigrants from outside Hampden County		
	Algorithm I	Algorithm II	Census estimate
Agawam	26.0	5.6	1.7 - 2.6
Easthampton	1.4	0.37	
East Longmeadow	17.0	1.8	1.4 - 2.1
Granby	N.C	0.66	
Hampden	0.07	11.4	0.9 - 1.2
Longmeadow	1.7	17.0	0.9 - 1.8
Ludlow	0.85	0.4	1.0 - 2.8
South Hadley	0.64	12.0	
Southampton	0.64	0.14	
Southwick	4.5	0.1	
Westfield	1.33	0.28	0.4 - 2.4
West Springfield	0.75	0.62	1.2 - 3.7
Wilbraham	0.26	N.C	1.5 - 3.6

N.C. = no convergence within the specified smoothing domain

Table 10 Estimated ratios of inmigrants for selected three-region systems.

	Estimated ratio of out-migrants (see text)
Agawam	12.0
Easthampton	3.1
East Longmeadow	12.5
Granby	N.C
Hampden	N.C.
Longmeadow	45.0
Ludlow	1.8
South Hadley	15.0
Southampton	2.6
Southwick	7.2
Westfield	0.6
West Springfield	7.5
Wilbraham	5.3

N.C. = No convergence within specified smoothing domain

Table 11 Estimated ratios of outmigrants for selected three-region systems and in only isolated cases comparable with the ratios estimated from the Census.[9] Table 11 shows the ratio of outmigrants (from a given town) to areas outside Hampden County to outmigrants to other parts of Hampden County, obtained from the third row of the estimated growth operator for the three-region systems Massachusetts-Hampden County-Suburban town. Again the results are erratic, and in some cases of improbable magnitude.

It may be concluded that the extension of the computational model to three-region systems requires the use of additional symptomatic indicators of intercensal population, since the results obtained by using only vital statistics appear unsatisfactory. Nevertheless, the quality of the results for two-region systems would justify additional research into multi-region systems using further symptomatic variables. The two-region formulation suffices for the stochastic population projection models of succeeding chapters; anticipated future refinements to such projection models will undoubtedly demand a more detailed quantification of intra- and interregional migration streams.

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[9] The census estimates can again only be specified as a range; see previous section.

CHAPTER VI

TIME SERIES ANALYSIS OF BIRTH, DEATH AND MIGRATION RATES

Birth and Death Rates

Autoregressive formulation. Figures 17 and 18 show birth and death rates for a selection of communities in the Lower Pioneer Valley Regional Planning District. Births and deaths were obtained from the allocated vital statistics records published by the Commonwealth of Massachusetts, and the intercensal population is interpolated by the method of Chapter V.

In view of the apparent periodicities, one line of approach to a stochastic formulation of vital rates would be a Fourier Series decomposition and subsequent analysis by spectral and cross-spectral methods after application of an appropriate filter to eliminate trend. A recent paper by Coale (51) has explored this approach in the analysis of fertility cycles in Sweden. However, the underlying assumption that fertility variations are indeed cyclical becomes questionable when dealing with small open populations. For variations in mortality a cyclical decomposition would seem equally inappropriate. At a local level, variations in vital rates are more likely to occur in response to differentials in socio-economic composition and rapid changes in age structure resulting from age-selective migration movements than in sympathy with cycles on a national level.

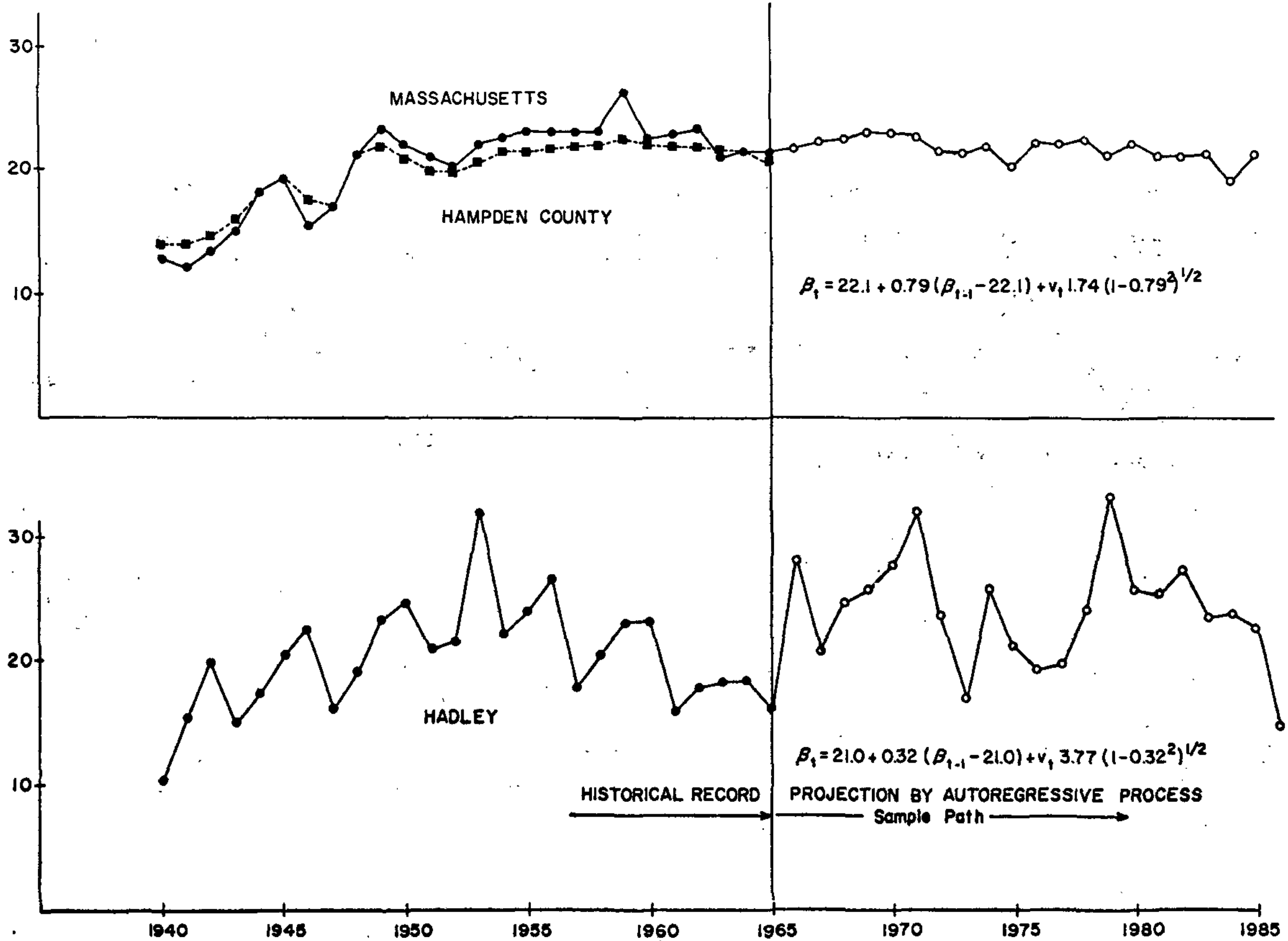

Figure 18 : birth reat time series for Hadley, Hampden County and Massachusetts, 1940-1985

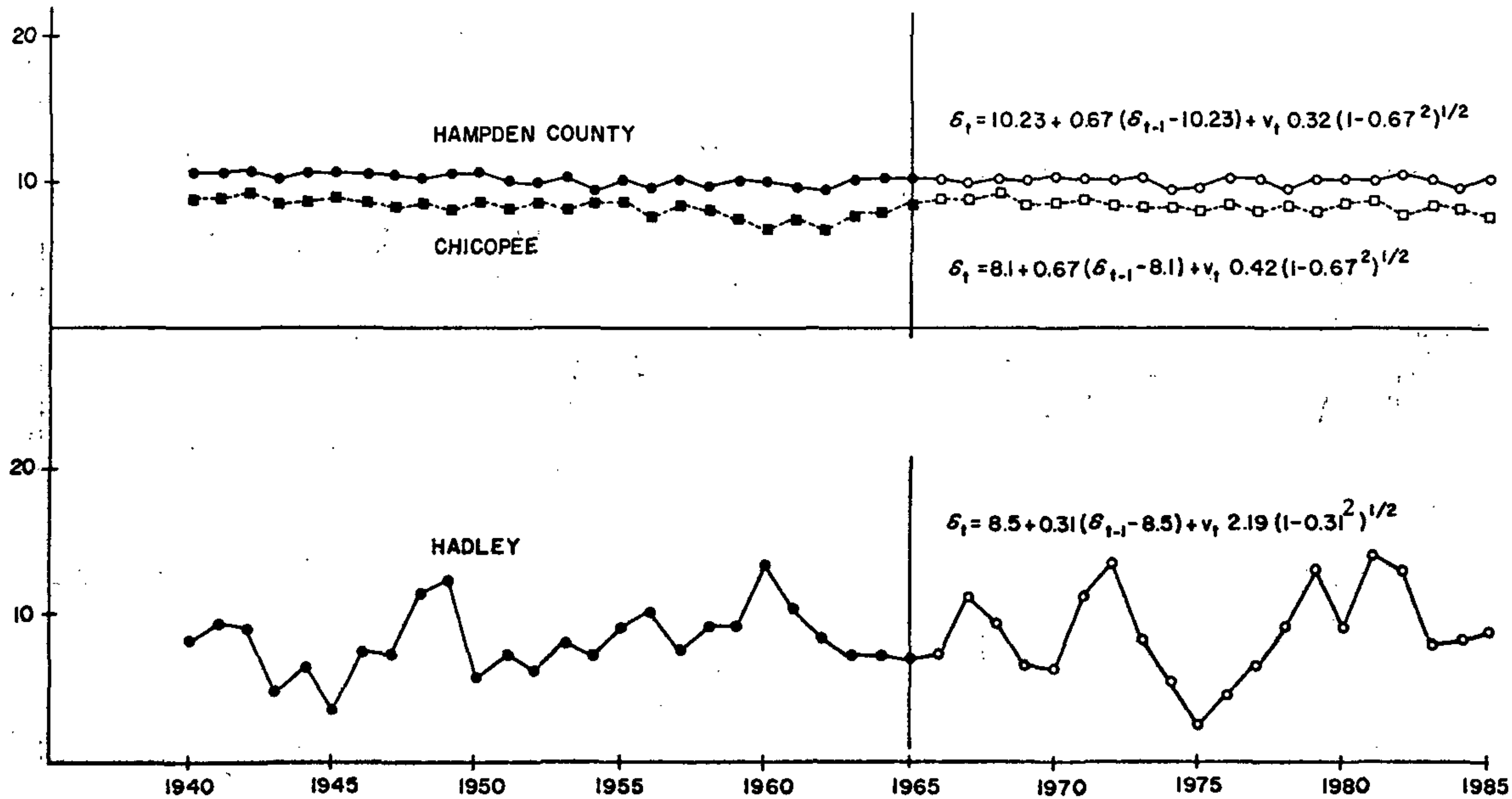

Figure 17 : Death rate time series for Hadley, Hampden County and Chicopee, 1940-1985

Inspection of Figures 17 and 18 suggests an inverse relation between the size of a community or region and the magnitude of the fluctuations about the mean. If the series is regarded as stationary over limited time periods, then a convenient formulation of the underlying stochastic process is the n-th order linear autoregressive model

$$\beta_i^t = a_o + a_1 \beta_1^{t-1} + a_2 \beta_i^{t-2} + \ldots + a_n \beta_i^{t-n} + u^t \quad [6.1]$$

$$\delta_i^t = a_o' + a_1' \delta_1^{t-1} + a_2' \delta_i^{t-2} + \ldots + a_n' \delta_i^{t-n} + u^t \quad [6.2]$$

where a_i = autoregression coefficients

u_i^t = random terms

β_i^t = birth rate of region i at time t

δ_i^t = death rate of region i at time t

This formulation accounts for the persistence of deviations from trend, according to the magnitude of the serial correlation coefficients. By fitting such a relation to observed time series, it is then possible to obtain a quantitative relation between the magnitude of the random components and serial correlations and population size. However, since the series may not be strictly stationary over the full period considered, the estimated mean may vary accordingly. For example, computations for the series 1940-1965 and 1950-1965 show significant differences in the estimated mean, but also show insignificant differences in the subsequent relationships between the magnitude of the random term and serial correlation coefficients and population size.

Since the means do vary with the length of record considered, it is imprudent to assume stationarity into the future. The sample paths of Figures 17 and 18 (representing one possible manifestation of the process) are therefore for illustrative purposes only. However, we may utilize the estimated means for a particular time period in a cross-section analysis in order to explain variations in those means in terms of pertinent socio-economic and demographic variables.

Statistical problems. As we have noted in Chapter III, application of least squares to an autoregressive time series may result in significant small sample bias. The Monte Carlo study by Orcut and Winokur (52) has demonstrated, however, that standard least squares prediction is no less optimal than some of the corrected estimators that have been suggested previously (see for example Marriott and Pope (53) or Quenouille (54)). The earlier study by Orcut and Cochrane (55) showed that in the presence of autocorrelated residuals, the assumptions underlying the Gauss-Markov theorem break down completely (Goldberger (27)), yielding estimators that are inconsistent. Unfortunately, the Durbin-Watson test for the detection of autocorrelated regression residuals has low power in an autoregressive model, and the distributions of non-parametric tests available for testing non-randomness in a series have not yet been derived for the case of residuals from a fitted regression. In view of the excellence of fit of the estimated parameters on the set of variables for which we have good prior grounds as being explanatory

	Births			Deaths		
	$\hat{\bar{\beta}}$	$\hat{\rho}(\beta)$	$\hat{\sigma}(\beta)$	$\hat{\bar{\delta}}$	$\hat{\rho}(\delta)$	$\hat{\sigma}(\delta)$
Springfield	23.23	0.72	1.41	11.26	0.66*	0.49
Holyoke	21.86	0.77*	1.81	12.87	0.37*	0.66
Chicopee	25.15	0.83*	1.85	8.06	0.67*	0.67
Agawam	19.36	0.64*	1.98	7.24	0.46*	1.14
Amherst	19.48	0.65*	2.64	8.71	0.47*	1.66
Belchertown	13.14	0.51*	2.46	7.71	0.33	1.43
Easthampton	19.93	0.66*	2.28	9.66	0.01	1.02
East Longmeadow	17.51	0.59*	2.97	7.76	0.59*	1.47
Granby	22.15	0.68*	4.69	6.88	0.24	2.36
Longmeadow	11.95	0.15	1.88	7.49	-0.13	1.37
Ludlow	20.97	0.69*	2.12	7.27	0.09	0.80
Northampton	17.48	0.77*	1.31	10.25	0.26	0.79
South Hadley	22.03	0.83*	2.07	8.43	0.31	1.16
Southwick	23.07	0.75*	3.55	7.89	0.52*	2.25
W.Springfield	21.42	0.76*	1.98	8,88	0.07	0.75
Westfield	21.13	0.74*	1.88	10.44	-0.19	0.80
Wilbraham	17.77	0.26	3.05	7.72	0.41*	1.73
Blandford	20.72	0.36*	4.78	11.07	0.02	4.50
Chester	18.33	-0.11	4.69	12.76	0.22	3.82
Granville	19.97	0.38*	6.65	12.67	0.38*	3.75
Hadley	20.99	0.32*	3.77	8.53	0.31	2.19
Hampden	17.61	0.14	5.12	8.38	0.19	3.23
Huntington	20.45	0.25	5.00	13.21	0.08	2.73
Middlefield	19.26	0.02	8.10	10.58	0.06	5.76
Montgomery	19.33	0.04	8.67	8.67	-0.02	7.05
Pelham	18.61	0.05	6.11	10.18	0.05	4.56
Russel	21.49	0.14	5.20	10.33	-0.19	2.60
Southampton	17.58	8.19	3.08	8.72	0.23	2.97
Tolland	10.69	-0.13	10.37	10.04	0.13	10.86
Westhampton	23.61	0.19	7.51	9.72	-0.06	5.56
Hampden County	22.06	0.79*	1.74	i0.23	0.34*	0.32
Massachusetts	21.25	0.80*	1.19	10.92	0.67*	0.30

(*) denotes significance (reject null hypothesis) at α=0.05

Table 12 First-order autoregressive parameters for birth and death rates for communities in the LPVRPD over the interval 1940-1965

	Births			Deaths		
	$\hat{\bar{\beta}}$	$\hat{\rho}(\beta)$	$\hat{\sigma}(\beta)$	$\hat{\bar{\delta}}$	$\hat{\rho}(\delta)$	$\hat{\sigma}(\delta)$
Springfield	24.33	0.53*	1.5	11.8	0.34	0.4
Holyoke	21.96	0.13	1.25	12.88	0.15	0.62
Chicopee	30.10	0.71*	1.6	9.27	0.24	0.43
Agawam	29.2	0.78*	2.8	10.20	0.05	1.41
Amherst	25.7	0.58*	2.75	9.86	0.58*	1.05
Belchertown	17.6	-0.15	2.41	8.62	0.18	1.35
Easthampton	22.5	0.56*	2.1	10.5	-0.23	1.24
East Longmeadow	30.8	0.85*	6.4	12.5	0.38	1.93
Granby	55.5	0.31	10.0	13.1	-0.11	4.1
Longmeadow	17.8	0.54*	2.7	10.42	-0.03	2.3
Ludlow	33.1	0.78*	2.7	9.88	0.49*	1.18
Northampton	18.7	0.48*	1.1	10.54	0.35	1.0
South Hadley	31.4	0.82*	3.2	11.53	0.18	1.1
Southwick	27.9	0.96*	5.9	11.86	0.09	3.2
West Springfield	25.7	0.68*	1.54	10.39	-0.18	0.66
Westfield	25.6	0.78*	1.31	12.06	-0.03	1.08
Wilbraham	29.4	0.22	4.5	11.65	-0.06	2.1
Blandford	27.3	0.29	4.60	13.04	0.22	5.1
Chester	19.2	-0.32	4.50	12.55	0.14	4.3
Granville	26.5	0.52*	6.70	13.46	0.32	4.05
Hadley	24.7	-0.01	4.25	10.10	0.49*	2.05
Hampden	32.6	0.55*	9.20	11.81	-0.28	3.6
Huntington	23.28	-0.14	5.00	14.3	0.19	3.2
Middlefield	26.59	0.12	10.20	14.4	0.20	5.8
Montgomery	34.5	0.55*	14.00	15,77	0.31	12.7
Pelham	27.8	0.11	6.87	11.35	-0.11	0.2
Russel	25.45	-0.36	4.20	10.57	-0.07	1.78
Southampton	28.1	0.32	5.20	11.75	-0.42*	3.77
Tolland	12.73	-0.24	10.70	8.9	0.09	8.7
Westhampton	30.4	0.18	7.90	11.21	0.42*	4.8
Hampden County	25.5	0.51*	1.74	11.39	0.33	0.36
Massachusetts	23.4	0.83*	0.51	11.69	-0.03	0.05

(*) denotes significance (reject null hypothesis) at α=0.05

Table 1[3] First-order autoregressive parameters for birth- and death rates for communities in the LPVRPD over the interval 1950-1965 (for example the magnitude of the random term as a function of population size), significant bias, if any, appears to be uniform over the entire set of communities. The numerical results are presented fully cognizant of their limitations.

Serial correlation and the random term. Both first and second order schemes were fitted to the vital rates of all thirty towns in the Lower Pioneer Valley Regional Planning District by use of least squares. Since the results of the second order scheme were not demonstrably better than those of the first order scheme, attention was focused on the latter.[1]

The first order model was fitted to two series, 1940-1965 and 1950-1965, and the resultant parameters are tabulated on Tables 12 and 13. In view of the rise in birth rates in the late Forties, the estimated means were significantly higher for the latter series. There were also fewer significant serial correlations for the 15-year series (20 as against 31 for the 25-year series), which again reflects the desirability of utilizing as long a series as possible.

By an analogous argument to that developed in Chapter IV for serially correlated migration rates, the mean level of the first-order process

$$\beta_i^t = a_o + a_1 \beta_i^{t-1}$$

is given by

---

[1]The estimated coefficients for the second-order scheme are tabulated in Appendix C.

$$\hat{\bar{\beta}}_i = \hat{a}_o(1 - \hat{a}_1) \quad \text{[6.3]}$$

and the serial correlation and random term variance by

$$\hat{\rho}_i(\beta) = \hat{a}_1 \qquad \hat{\sigma}_i(\beta) = u^t/[v^t\ (1 - a_1^{(2)})^{(0.5)}]. \quad \text{[6.4]}$$

Regressing the estimated serial correlation coefficients and the estimated variance of the random term on average population over the interval yielded

$$\hat{\rho}_i(\beta) = 0.007\ p^{0.278} \qquad R^2 = 0.63 \quad \text{[6.5]}$$

$$\hat{\rho}_i(\delta) = 0.015\ p^{0.443} \qquad R^2 = 0.58 \quad \text{[6.6]}$$

$$\hat{\sigma}_i(\beta) = 98.4\ p^{-0.467} \qquad R^2 = 0.97 \quad \text{[6.7]}$$

$$\hat{\sigma}_i(\delta) = 47.1\ p^{-0.313} \qquad R^2 = 0.91 \quad \text{[6.8]}$$

where $\hat{\rho}_i(\beta)$ = estimated birthrate serial correlation coefficient, region i

$\hat{\rho}_i(\delta)$ = estimated deathrate serial correlation coefficient, region i

$\hat{\sigma}_i(\beta)$ = standard deviation, random component of birthrate process, region i

$\hat{\sigma}_i(\delta)$ = standard deviation, random component of deathrate process, region i

p = average population over the interval 1940-1965

Mean birth and death rates. The mean levels of the autoregressive processes (mean birth and death rates) were analysed in terms of regional deviations. It was hypothesized that deviations from the regional means could be explained in terms of deviations in explanatory variables, especially differentials in age structure and income. The significance of this type of formulation will become apparent in Chapter VII where the here-established relationships will be used for projection purposes. Introducing the notation

$\hat{\beta}_i$ = estimated mean birth rate, town i

$\hat{\beta}_T$ = estimated mean birth rate for entire region (LPVRPD)

$\hat{\delta}_i$ = estimated mean death rate, town i

$\hat{\delta}_T$ = estimated mean death rate, entire region

P_{ji} = fraction of individuals in j-th age group, town i

P_{jT} = fraction of individuals in j-th age group, entire region

Q_i = per household income, town i

Q_T = per household income, entire region

The age groups are defined as follows:

j = 1 0-14 years of age

j = 2 15-24 years of age

j = 3 25-44 years of age

j = 4 45-65 years of age

j = 5 greater than 65 years of age

Regressing the mean birth and death rates on P_{ij}, j = 1, 2, ... 5 and Q_i resulted in the following relationships (where insignificant explanatory variables have been omitted):

$$\frac{\hat{\hat{\delta}}_i}{\hat{\hat{\delta}}_T} = \frac{P_{5i}}{P_{5T}}^{0.43} \frac{P_{4i}}{P_{4T}}^{0.09} \frac{Q_i}{Q_T}^{-0.3} \quad . \; . \; . \; . \; . \; . \; . \; [\ 6.9\ ]$$

$R^2 = 0.85$

$$\frac{\hat{\hat{\beta}}_i}{\hat{\hat{\beta}}_T} = \frac{P_{2i}}{P_{2T}}^{0.17} \frac{Q_i}{Q_T}^{-0.821} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 6.10\ ]$$

$R^2 = 0.97$

Conclusions. Eq.[6.5] - [6.8] confirm the relationship between population size and random term and serial correlation coefficient of the autoregressive model for vital rates. The dependence of deviation from the regional deathrate in terms of differential age structure and income is not unexpected, with the younger age groups not significant in the corresponding regression equation (Eq.[6.9]). The significant variables in Eq.[6.10] that explain deviations from the regional birth-rate are also consistent with expectations.

Migration Rates

Formulation: Several regression models of interstate migration have appeared in recent years (see for example Rogers (56), Lowry (57) or Greenwood (58)). However, the explanatory variables in the context of interstate or inter SEA (State Economic Area) flows, for example unemployment rates, income and existing migrant stock at the place of destination, climate, education, are of a different nature than those applicable to intra-regional migrations within an urban area. The problem in our context is to estimate the resulting distribution of residential location given some rates of in- and outmigration for the region as a whole. The explanatory variables used in the multiple regressions to establish these latter relationships are listed on Table 14. Again the analysis is in terms of deviations from the regional mean, since we seek to explain differentials in growth and migration rates in terms of differentials in a set of explanatory variables that are less difficult to project into the future.

Results. The results of regressing the 3 dependent variables of Table 14 on the entire set of independent variables given on Table 14 are shown on Table 15, where the insignificant variables are again omitted. A positive exponent indicates a positive contribution of the corresponding independent variable to the growth or migration rate, a negative exponent indicates an inverse relationship.

Some observations on the coefficients of Eq.[6.11] - [6.13] are in order. It is of interest to note that sewer access

Independent (Explanatory) Variables	
X_1	Distance to the central city of the SMSA, miles
X_2	Fraction of population without sewer access, 1965
X_3	Fraction of population without public water supply
X_4	Single family housing as a fraction of total residential land use, 1965
X_5	Vacant acres, 1965
X_6	Acres of water bodies, wetlands, forest, 1965
X_7	Year of enactment of first zoning law
X_8	Taxrate, 1965 (adjusted for variable assessment ratio)
X_9	Net residential density, 1965
X_{10}	Gross population density, 1965
Dependent Variables	
Y_1	Total inmigrants, 1955-1960 (as obtained from Algorithm II, chapter IV)
Y_2	Total outmigrants, 1955-1960 (as obtained from Algorithm II, chapter IV)
Y_3	Net population increase, 1955-1965

Table 14 Definition of variables for the multiple regression models of growth and migration rates.

Table 15 Estimated growth relationships for 13 suburban towns

Relative Growth Rates

$$\frac{Y_{3i}}{Y_{3T}} = 0.29 \left(\frac{X_{2i}}{X_{2T}}\right)^{-0.13} \left(\frac{X_{4i}}{X_{4T}}\right)^{6.69} \left(\frac{X_{5i}}{X_{5T}}\right)^{0.52} \left(\frac{X_{9i}}{X_{9T}}\right)^{0.99} \quad \ldots \text{[6.11]}$$

Sewer access; SF housing; Vacant Acres; Residential Density

$R^2 = 0.87$

Relative inmigration Rates

$$\frac{Y_{1i}}{Y_{1T}} = 0.36 \left(\frac{X_{2i}}{X_{2T}}\right)^{-0.12} \left(\frac{X_{4i}}{X_{4T}}\right)^{4.75} \left(\frac{X_{5i}}{X_{5T}}\right)^{0.33} \left(\frac{X_{6i}}{X_{6T}}\right)^{0.23} \left(\frac{X_{9i}}{X_{9T}}\right)^{1.34}$$

Sewer access; SF housing; Vacant acres; Forest, wetlands; Residential density

[6.12]

$R^2 = 0.97$

Relative Outmigration Rates

$$\frac{Y_{2i}}{Y_{2T}} = 0.59 \, X_{1i}^{-0.59} \left(\frac{X_{4i}}{X_{4T}}\right)^{-7.09} \left(\frac{X_{5i}}{X_{5T}}\right)^{0.27} \left(\frac{X_{6i}}{X_{6T}}\right)^{0.31} \left(\frac{X_{7i}}{X_{7T}}\right)^{0.56}$$

Distance to CC; SF housing; Vacant Acres; Forest, Wetlands; Zoning Law

$$\left(\frac{X_{8i}}{X_{8T}}\right)^{1.46} \left(\frac{X_{9i}}{X_{9T}}\right)^{0.66} \quad \ldots \text{[6.13]}$$

Taxrate; Res.density

$R^2 = 0.95$

appears as a positive inducement to both overall growth and to inmigration[2]. As expected, the character of residential neighbourhoods as measured by the fraction of single family homes is significant as an inducement for inmigration, but equally significant as a determinant for out-migration (negative coefficient for X_4 in Eq.[6.13]). In part, this reflects the fact that residence times in single family homes are of greater duration than in multi-family housing, and confirms the intuitive and often invoked reasoning that appartment complexes are detrimental to the"stability" of a town. A higher than average taxrate apparently acts as an incentive for out-migration, but not as a disincentive for in-migration, since this variable is not significant in Eq.[6.12]. The presence of X_5 (fraction of total vacant land in the i-th region) in Eq.[6.11] and [6.12] is anticipated, reflecting the dependence of residential growth on construction opportunities. Of intriguing interest is the presence of X_7 in the out-migration equation. A low value of X_7 is indicative of a long-established zoning regulation; the positive coefficient is thus indicative of an incentive to leave a relatively poorly planned community.

The reader is reminded that the particular set of variables found significant, and the magnitude and sign of their associated coefficients are specific to the set of towns here examined. These equations will be utilized in the following chapter for population projections of the same set of towns. Application of the methodology developed in these chapters to a different region would require re-estimation of these equations.

Since many of the variables of Table 14 are not available on a time series basis (for the LPVRPD there exists *only the* one land-use study prepared in 1965), one is limited to a single cross-section analysis to establish the growth and migration correlates of the type here considered. That the dependence of migration rates on particular variables may change in future times is fully recognized. Nevertheless, even a single cross-section analysis represents a significant advance over intuitive guesswork as to the probable future development of growth rates.

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[2] In order to avoid zeros in the logarithmic transformation of the independent variables, variables X_2 and X_3 needed to be formulated as the fraction without sewer access and public water supply; A negative regression coefficient implies a positive relation to the fraction with access.

CHAPTER VII

POPULATION PROJECTIONS FOR LOCAL AREAS

Applications of stochastic process theory. A central theme of this study has been the stochastic nature of demographic events and the emphasis on migration as a principal determinant of population change in a local area. A review of the literature of stochastic process theory, however, reveals relatively little work on human population growth subject to migration.

Several authors have attempted to extend the classical birth-and-death models of Yule (59) and Kolmogorov (60) to incorporate migration. In general, explicit mathematical expressions for the stochastic mean can be obtained by the use of generating function techniques for the solution of the governing Kolmogorov differential equations. In certain cases, explicit expressions for the variance and individual state probabilities have also been obtained.

For example, the mean and variance of a stochastic birth-and death process are given by

$$E\{X(t)\} = X(0)\ e^{(\beta - \delta)t}$$
$$\mathrm{Var}\{X(t)\} = X(0)\ \frac{\beta+\delta}{\beta-\delta}\ e^{(\beta-\delta)t}\ (e^{(\beta-\delta)t} - 1) \qquad [\ 7.1\ ]$$

where β = infinitesimal birth rate

δ = infinitesimal death rate

$X(0)$ = number of individuals in system at time zero

$X(t)$ = number of individuals in system at time t

We note that the stochastic mean is equal to the corresponding deterministic value.[1] Introduction of a constant inmigration rate μ results in the following modification to Eq.[7.1] (Bailey (61))[2]:

$$E\{X(t)\} = \frac{\mu}{\beta-\delta}\{e^{(\beta-\delta)t} - 1\} + X(0)\, e^{(\beta-\delta)t} \quad . \; . \; . \; [\ 7.2\ ]$$

As soon as we permit births and deaths to become functions of time, solutions for the moments involve the evaluation of integral equations. Bailey (61) shows that for the simple non-homogenous birth-and-death process

$$E\{X(t)\} = X(o)\, e^{-\rho(t)}$$

$$\rho(t) = \int_0^t \{\delta(\tau) - \beta(\tau)\}\, d\tau \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 7.3\ ]$$

De Cani (62) considered the simultaneous growth of two populations linked by migration. Again using the moment generating function technique, several pages of "brute force" mathematics yield for the expected value

$$E\{X_1(t)\} = \frac{\mu_2}{\mu_1+\mu_2}(X_1(0) + X_2(0))\, e^{(\beta_1-\delta_1)t}$$

$$+ \frac{\mu_1 X_1(0) - \mu_2 X_2(0)}{\mu_1 + \mu_2}\, e^{(\beta_1-\delta_1-\mu_1-\mu_2)t} \quad . \; . \; [\ 7.4\ ]$$

---

[1] The corresponding deterministic formulation is given by

$$\frac{dX(t)}{dt} = (\beta-\delta)\, X(t)$$

for which integration yields the well known result $X(t)=X(0)e^{(\beta-\delta)t}$

[2] In the literature of stochastic processes, the symbols λ and μ are conventional for birth and death rates, respectively, and m is conventional for migration rates. For consistency of notation, however, we shall continue to designate birth, death and migration rates as β,δ and μ, respectively.

where μ_1 and μ_2 represent the migration rates from region 1 to region 2 and region 2 to region 1 respectively. Under the simplifying assumption

$$\mu_1 = \mu_2 = \mu$$
$$\beta_1 = \beta_2 = \beta$$
$$\delta_1 = \delta_2 = \delta$$
$$\alpha = \beta - \delta$$

De Cani obtained for the variance the expression

$$\mathrm{Var}\{X_1(t)\} = \frac{X_1(0) + X_2(0)}{2} e^{2\alpha t} \left[\frac{\beta+\delta}{2\alpha} + \frac{\beta+\delta+4}{2(\alpha-4\mu)} e^{-4\mu t} - \frac{(\beta+\mu)\alpha - 4\mu\delta}{(\alpha-4\mu)} e^{-\ t}\right] + \frac{X_1(0)-X_2(0)}{2}\left(\frac{\beta+\delta}{\alpha}\right) e^{2(\alpha-\mu)t} (1 - e^{-\alpha t}) \quad . \ . \ . \ . \ . \ . \ . \ . \ [\ 7.5\ ]$$

from which the complexity of the corresponding non-homogenous case can be anticipated. De Cani's two-population model appears to be the only multi-dimensional process described in the literature set in the context of interregional migration, although multi-dimensional models have attained formidable erudition in genetics.

Adke and Moyal (63) developed a model of population growth whose individuals are subject to diffusion along a line in addition to undergoiung births and deaths. Given an initial population at the origin, the expression for the mean is asymptotically Gaussian. Adke (64) generalized this result to the non-homogenous case. In a more recent development, motivated by the implications for cancer research, Bailey (65) considered the growth of a set of populations each located at the nodes of a square lattice, each subject to births, deaths and migrations to and from adjacent nodes. The first moment is given by

$$E\{X_{ij}(t)\} = e^{-(\beta-\delta-\mu)t} \sum_k \sum_\ell X_{k\ell}(0)\ I_{i-k}(\tfrac{\mu t}{2})\ I_{j-\ell}(\tfrac{\mu t}{2}) \quad [\ 7.6\ ]$$

where $X_{ij}(t)$ is the population at points i,j of the lattice at time t and I_j is a Bessel function of the first kind. For the special case of a single initial colony of size $X_a(0)$ at the origin at time t=0,

$$E\{X_{ij}(t)\} = X_a(0)\ e^{-(\beta-\delta-\mu)t}\ I_i(\tfrac{\mu t}{2})\ I_j(\tfrac{\mu t}{2}) \quad .\ .\ .\ .\ [\ 7.7\ ]$$

In summary it appears that the present results of stochastic process theory offer little assistance to the planner upon whom falls the burden of a quantitative population projection. The analytical complexities of the general multi-dimensional non-homogenous process that is required to adequately describe human population growth preclude, at present, direct application. Nevertheless, available results do afford considerable insight to the nature of stochastic growth processes.

Matrix Methods. In the study of population dynamics increasing use ha been made of the basic matrix formulation

$$\underset{(k \times 1)}{w^{t+1}} = \underset{(k \times k)}{\Phi} \; \underset{(k \times 1)}{w^{t}} \quad \ldots\ldots\ldots\ldots \quad [\ 7.8\ ]$$

to describe a population distribution at two succeeding points in time. Formulated originally by Leslie (66), and expanded by Leslie (67), Keyfitz (68), Pollard (69), Lopez (70) and Sykes (71), among others, the k components of w^t have normally represented the number of individuals in each of the quinquennial age groups 0-4, 5-9, ... etc. The (k x k) matrix Φ characterises the transformation from w^t to w^{t+1}. Φ is thus a matrix of survival ratios ${}_i s_{i+1}$ representing the proportion of individuals of the i-th age group who survive to become members of the (i+1)-st age group at the end of the unit time period, and of birth rates β_i, representing the number of births that survive to the end of the time interval born to the i-th child bearing age group. Thus

$$\Phi = \begin{bmatrix} 0 & 0 & \beta_1 & \beta_2 & . & . & . & 0 \\ {}_1s_2 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & {}_2s_3 & 0 & 0 & . & . & . & 0 \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & 0 & . & . & {}_{n-1}s_n & 0 \end{bmatrix} \quad \ldots \quad [\ 7.9\ ]$$

It is evident that this matrix model may be used as a projection device, and mathematical erudition has not been lacking in the development of such models. However, they are generally restricted in application to closed populations, undisturbed by migration. This restriction is no hindrance to the study of national populations, for which the migration component of growth may either be neglected (at least numerically, if not by social or occupational group), or where political constraints (immigration quotas etc.) permit realistic estimates of future migration.

Sykes (71) appears to contain the first analysis of the cohort-survival matrix model (Eq.[7.8]) as a stochastic process. In particular he considered the two models

$$w^t = \Phi\, w^{t-1} + u^t \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 7.10\ ]$$

which is analogous to the estimating equation [3.8] of Chapter III, and

$$w^t = \{\Phi + \Delta\}\, w^{t-1}$$

$$E\{\Delta_t\} = 0$$

$$E\{\Delta_{jk}\Delta_{lm}\} = \Sigma \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 7.11\ ]$$

which is in essence the model of Chapter IV, and derived analytically expressions for the mean and variance of the processes. As we have seen in earlier sections, the matrix formulation [7.8] was interpreted by Rogers as an interregional process, in which the elements of Φ represent no longer survival rates but interregional migration rates. The interregional population projections for the State Economic Areas of California (Rogers (57)) utilized Eq.[7.8] in this latter context. The analytical results obtained by Sykes (71) are not specific to the cohort-survival interpretation of the matrix model, and could be utilized in an interregional context. We have postulated, however, that the elements of Γ are serially correlated, and therefore the results of the process [7.11] cannot be directly utilized. The previous section has demonstrated that the mathematical treatment of non-homogenous stochastic processes is quite complex, for which analytical solutions are obtained at best with difficulty. We turn therefore to a simultaion approach that permits full consideration of the stochastic nature of demographic events in the framework of an interregional population projection without the complexities of analytical solutions.

Stochastic Simulation. With the widespread availability of time-sharing computer facilities, a stochastic simulation methodology provides a feasible alternative to the analytical approach for the evaluation of the projection moments by a regional planning agency. or their consulting engineer. Shubik (73) has defined simulation as follows:

> "... A simulation of a system or organism is the operation of a model or simulator which is a representation of that system or organism. The model is amenable to manipulations which would be impossible, too expensive or impractical to perform on the entity it portrays."

In the context of this study, the definition of simulation will be restricted to experiments on mathematical models. Analog and physical models are herein excluded.

Simulation models may be divided into two broad categories, deterministic and stochastic. Deterministic models have one predictable outcome for a given specified decision-situation, whereas stochastic models have a corresponding distribution of possible outcomes resulting from the inclusion of stochastic variables. This latter category is most commonly referred to in the literature as "Monte Carlo" simultaion,[1] though some authors have preferred the term "statistical simulation".[2] Another useful distinction is between dynamic and stochastic simulation.[3]

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[1]For example Fiering (38) or Hammersley and Handscomb (74)
[2]For example Eldredge (75) or Ehrenfeld and Ben-tuvia (76)
[3]A good introduction to computer simulation modelling is contained in Naylor et al. (77)

A static model is time-invariant, whereas a dynamic model includes time as an independent variable.[4]

The essential components of a stochastic simulation model comprise 1. A set of endogenous variables, specified entirely by the model itself.

2. A set of exogenous variables, specified a priori by the decision-maker. In the typical econometric model, such variables are those subject to policy decisions (government expenditure, taxation levels, etc.). In the demographic model, there exist also quasi-exogenous variables such as terminal regional fertility and mortality; these variables must be specified a priori, yet the planner has no direct control. Professional judgement will indicate what the possible range of such variables is liekly to be, and the results of sensitivity analyses will demonstrate their relative importance on the outcome.

3. A set of coefficients (or parameters), also specified a priori. The constants in Eq.[6.7] relating the magnitude of the random component of the birth and death processes to population size are examples of such coefficients.

4. A set of equations that describe the interrelationships between exogenous and endogenous variables, through time.

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[4]Smith's model of the wastewater treatment plant (19) is an example of a static, deterministic simulation, whereas Fiering's streamflow models are dynamic and stochastic (38). Forrester's model of a hypothetical urban system (78) and the Susquehanna River Basin Model (Hamilton et al., 79) are examples of dynamic, deterministic models.

In the illustrative development of the projection simulation model, we shall examine initially a closed single-region model subject only to births and deaths (Model A). Augmentation of Model A by net migrations represents the simplest viable projection technique (Model B). Finally, Model C will develop the full interregional projection model.

Stochastic Simulation Projections

Projection Model A. Model A, summarized in Table 16 and Figure 19, represents the basic birth-and-death sector for a given town, and will be incorporated into the multi-region models of subsequent sections. Population change by migration is temporarily excluded for the purpose of studying the basic properties of stochastic simulation projections.

A sample projection is defined by one application of the equations listed on Table 16, through time, with one particular drawing of random terms for the random variables. This procedure is repeated N times (N=sample size), each sample dependent on a new and different drawing of random terms, thus generating a set of projection curves and defining a probability distribution at each time point. From this distribution of sample values projection moments (mean, variance etc.) may be evaluated. A set of sample projections is illustrated on Figure 20, an and the distribution of sample values shown on Figure 21.

Introducing the notation

$\hat{x}^t_{ij}$ = j-th sample projection for region i, time t

$\hat{x}^t_i = \frac{1}{N} \sum_{j=1}^{N} \hat{x}^t_{ij}$ = expected projection for region i, time t based on a sample size N (=sample mean)

$s^t_i(x) = \frac{1}{N} \sum (\hat{x}^t_{ij} - \hat{x}^t_i)^{(2)}$ = standard deviation of sample outcomes, region i, time t

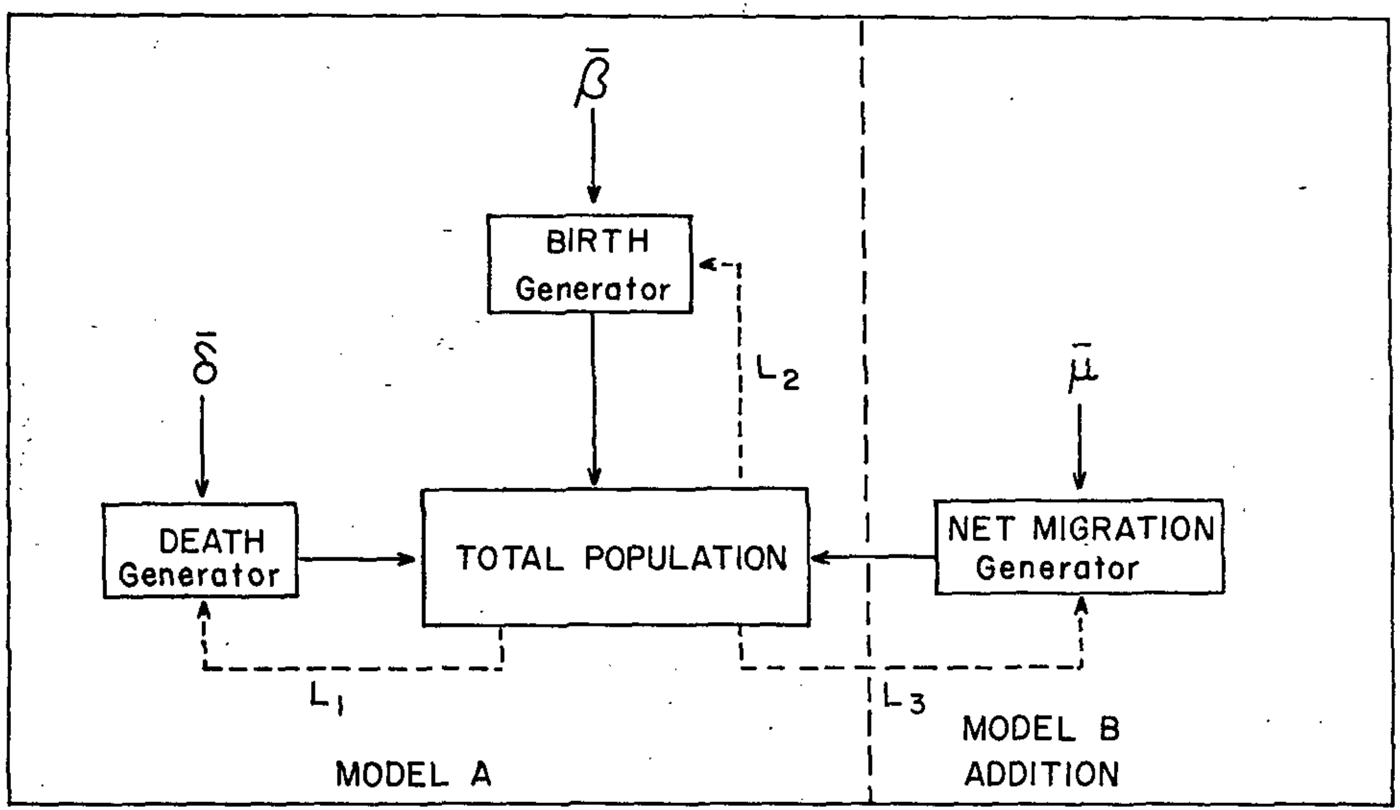


Figure 19 : Flow chart, population projection models A and B
(PROGRAM POPAB)

EQUATIONS

$$x^{t+1} = x^t (1 - \delta^t - \beta^t)$$

$$\beta^t = \overline{\beta} + \rho^t(\beta)\ (\beta^{t-1} - \beta) + v^t\ \sigma^t(\beta)\ [1 - \rho^t(\beta)^{(2)}]\mathrm{exp}\ 0.5$$

$$\delta^t = \overline{\delta} + \rho^t(\delta)\ (\delta^{t-1} - \delta) + w^t\ \sigma^t(\delta)\ [1 - \rho^t(\delta)^{(2)}]\mathrm{exp}\ 0.5$$

$$v^t = \mathrm{RND}(0,1) \qquad w^t = \mathrm{RND}(0,1)$$

$$\rho^t(\beta) = \rho^o(\beta)\ [\ \frac{x^t}{xo}\ ]\mathrm{exp}(a)$$

$$\rho^t(\delta) = \rho^o(\delta)\ [\ \frac{x^t}{xo}\ ]\mathrm{exp}(b)$$

$$\sigma^t(\beta) = \sigma^o(\beta)\ [\ \frac{x^t}{x^o}\ ]\mathrm{exp}(c)$$

$$\sigma^t(\delta) = \sigma^o(\delta)\ [\ \frac{x^t}{x^o}\ ]\mathrm{exp}(d)$$

EXOGENOUS VARIABLES

x^o = base year population

$\overline{\beta}$ = mean birthrate

β^o = base year birthrate

$\overline{\delta}$ = mean deathrate

δ^o = base year deathrate

PARAMETERS

a,b,c,d empirically determined coefficients (see Eq.[6.5 - 6.8])

ENDOGENOUS VARIABLES

x^t = population, time period t

β^t, δ^t = birth and death rates, time period t

$\rho^t(\beta)$, $\rho^t(\delta)$= serial correlation coefficients, time period t

$\sigma^t(\beta)$, $\sigma^t(\delta)$= standard deviation of random terms, period t

Table 16 : Projection Model A - Summary of Equations

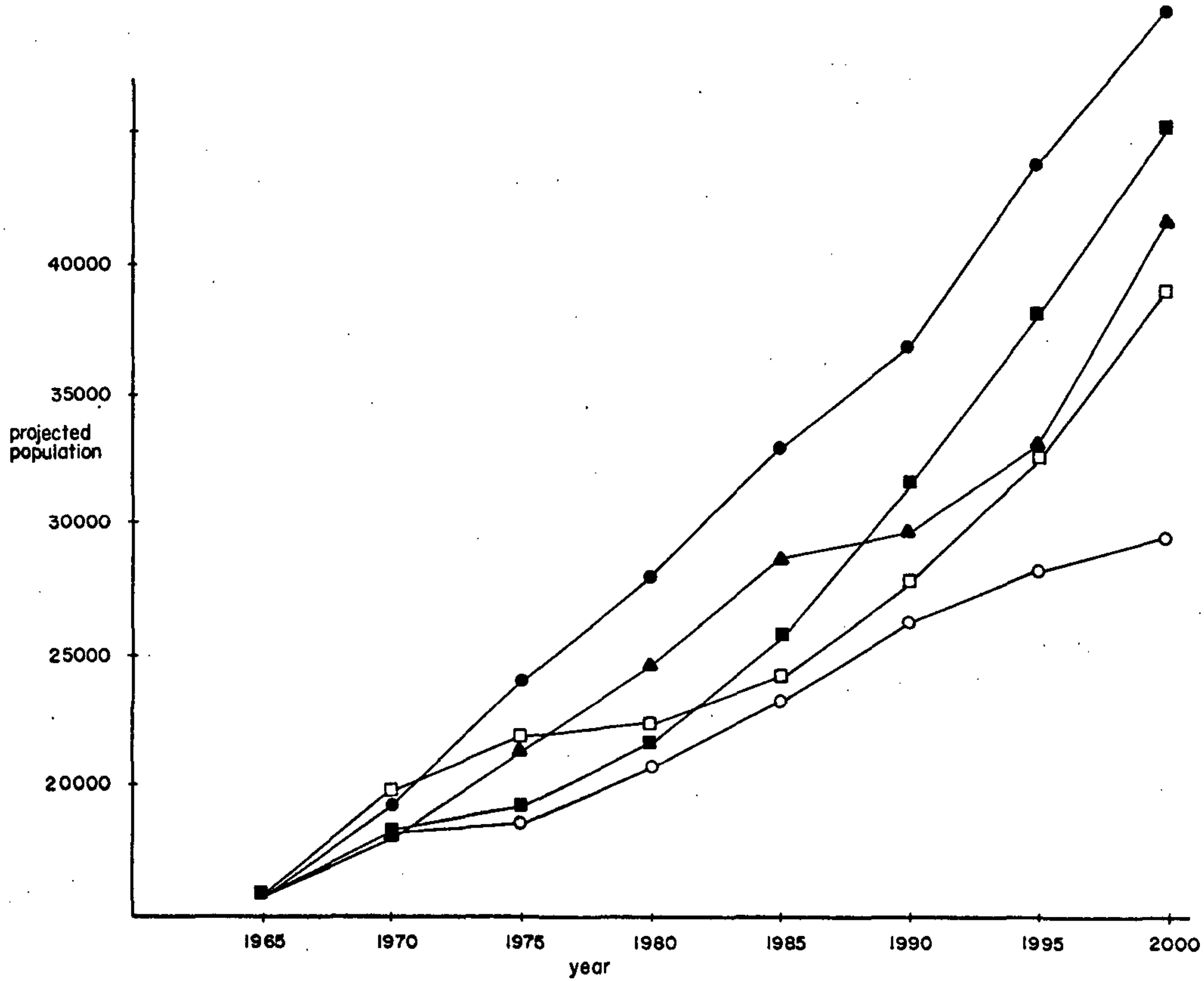


Figure 20 : Model A sample projections for Ludlow

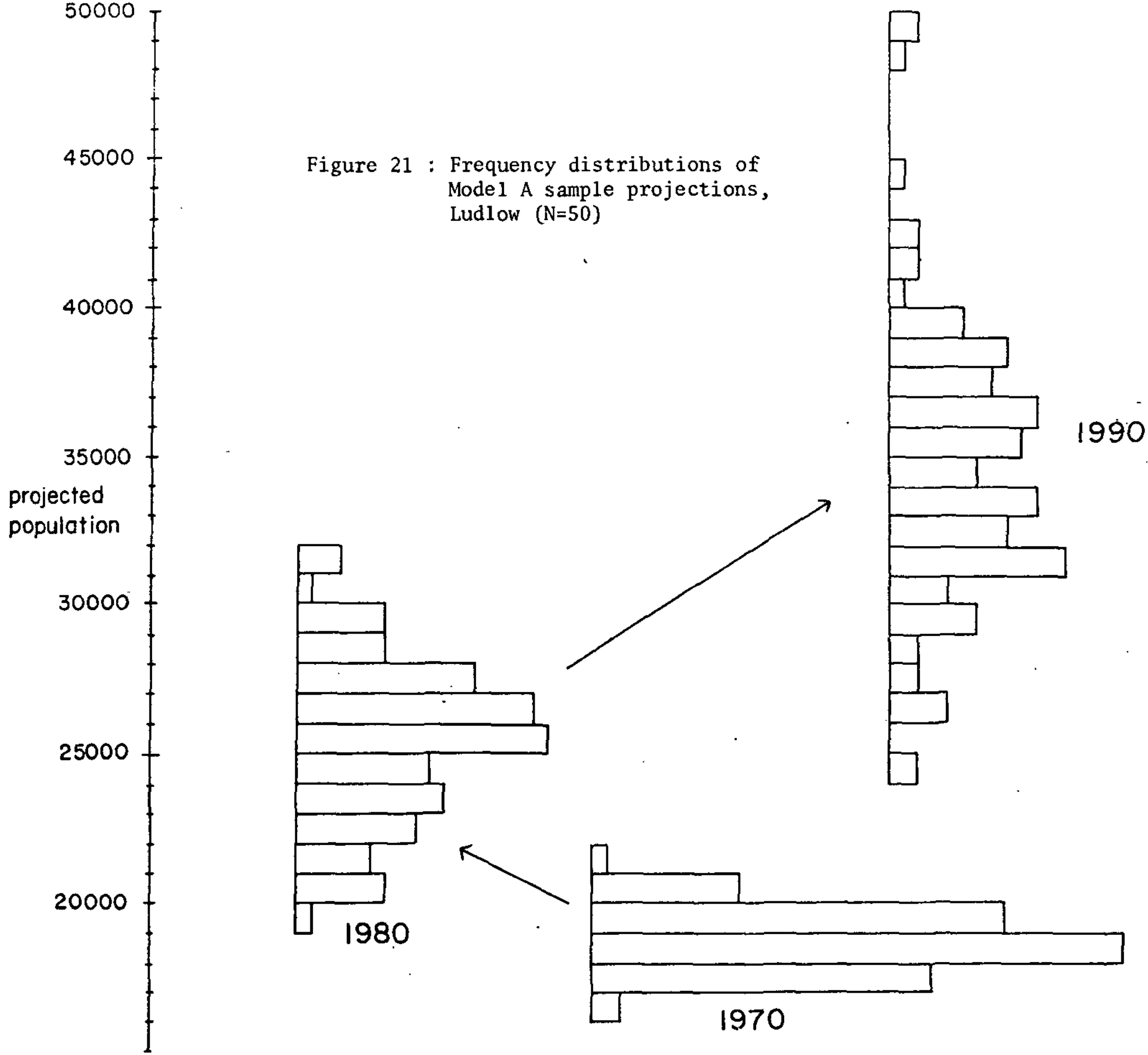


Figure 21 : Frequency distributions of Model A sample projections, Ludlow (N=50)

Figure 22

POPULATION PROJECTION LUDLOW

YEAR	ACTUAL POPULATION (PLOT CODE 1)	ACTUAL POP.DENS. (SQ.MI.)
1940	8181	287
1945	8065	283
1950	8660	304
1955	10530	369
1960	13805	484
1965	15922	559

YEAR	HIGH PROJECTION	LOW PROJECTION	EXPECTED PROJECTION PERMANENT	EXPECTED PROJECTION SEASONAL	POP DENSITY
1970	17380	16680	17040		598
1975	18880	17580	18240		640
1980	20440	18740	19600		688
1985	21940	20060	21000		737
1990	23520	21380	22400		788
1995	25400	22860	24140		847
2000	27580	23320	25940		911
2005	30340	25340	27840		977
2010	33140	26460	29800		1046
CODE	(*)	(*)	(P)		

```
  34000 +---------+---------+---------+---------+---------+---------+
        I                                                          *I
        I  *********************************                        I
        I  *                               *                   *    I
        I  *  PROJECTION MODEL A           *                       PI
  28600 +  *  AUTOREGRESSIVE BIRTH AND     *                        +
        I  *  DEATH PROCESS                *              *    P    I
        I  *  ZERO NET MIGRATION           *              P        *I
        I  *                               *         *         *    I
        I  *********************************         P    *         I
  23200 +                                       *    *              I
        I                                  *    P                   I
        I                                  P    *                   I
POP     I                             P*   *                        I
        I                        *    *                             I
  17800 +                        P*                                 +
        I                   P                                       I
        I              1                                            I
        I                                                           I
        I        1                                                  I
  12400 +                                                           +
        I                                                           I
        I    1                                                      I
        I1                                                          I
        I                                                           I
   7000 +---------+---------+---------+---------+---------+---------+
       1950      1960      1970      1980      1990      2000      2010
```

POPULATION PROJECTION BLANDFORD

YEAR		ACTUAL POPULATION (PLOT CODE 1)	ACTUAL POP.DENS. (SQ.MI.)
1940		479	9
1945		521	10
1950		597	11
1955		705	13
1960		636	12
1965		859	16

	HIGH PROJECTION	LOW PROJECTION	EXPECTED PROJECTION PERMANENT	EXPECTED PROJECTION SEASONAL	POP DENSITY
1970	940	860	900		17
1975	1000	880	940		18
1980	1060	900	980		19
1985	1120	940	1040		20
1990	1160	980	1080		20
1995	1220	1020	1120		21
2000	1280	1060	1180		22
2005	1360	1100	1240		23
2010	1460	1120	1300		24
CODE	(*)	(*)	(P)		

```
2000 +---------+---------+---------+---------+---------+---------+
     I                                                           I
     I  *********************************                        I
     I  *                               *                        I
     I  *  PROJECTION MODEL A           *                        I
1600 I  *  AUTOREGRESSIVE BIRTH AND     *                        +
     I  *  DEATH PROECESS               *                        I
     I  *  ZERO NET MIGRATION           *                       *I
     I  *                               *                   *    I
     I  *********************************              *    P   PI
1200 +                                       *    *    P         I
     I                                  *    P    P         *   *I
     I                        *    *    P         *    *         I
POP  I                   P    P    P    *    *                   I
     I              1         *    *                             I
 800 +                                                           +
     I     1                                                     I
     I          1                                                I
     I1                                                          I
     I                                                           I
 400 +                                                           +
     I                                                           I
     I                                                           I
     I                                                           I
     I                                                           I
   0 +---------+---------+---------+---------+---------+---------+
    1950      1960      1970      1980      1990      2000      2010
```

we may define a "high" and "low" projection as $\hat{x}_i^t + \pi s_i^t(x)$, $\hat{x}_i^t - \pi s_i^t(x)$ respectively, where π is such that the probability of the actual future population lying beyond these limits is Ω. On the assumption that the sample outcomes are distributed normally, we shall take $\Omega = 0.05$ and $\pi = 1.96$. If the model is an accurate representation of reality, one would expect the actual future population to fall within these limits with probability 0.95. In the projection of Figure 22 for the town of Ludlow, we hypothesize that there exists a 1 in 20 chance that the actual 1980 population is greater than 20440 or smaller than 18740.

A useful measure is the relative projection range, defined as

$$\bar{R}_p^t = \frac{R_p^t}{\hat{x}_i^t} = \frac{2\,\pi\, s_i^t(x)}{\hat{x}_i^t} \quad \text{. . . . . [7.12]}$$

A comparison of the relative projection range for 1980 between Blandford ($\hat{x}^{1980} = 980$, $\bar{R}_p^{1980} = 0.164$) and Ludlow ($R_p^{1980} = 0.087$) serves to illustrate the dependence of the projection variability on absolute population size.

<u>Projection model B</u>. The simplest modification to the birth and death process of Model A is to add a net migration component. In Massachusetts, net migration figures are available from Census data at 5 year intervals. Let $\ell_i^j(k)$ be the number of net migrants during the k-year interval commencing in year j for region i. Then we may define an average rate over the k-year interval as

Exogenous variables:

$\bar{\lambda}_i$ = mean net migration rate

$s_i(\lambda)$ = standard deviation of net migration rates

Endogenous Variable

λ_i^t = net migration rate in period t

to obtain the simplest viable projection model.

Application of this stochastic simulation projection to towns in the study region permits some interesting comparisons. Table 17 compares the 1980 probabilistic projection range R (column 1) with the range estimated by the planning consultant (consultant's "high" minus "low" projection"). Given that the stochastic model is an accurate representation of reality, and that the consultant's most probable projection lies midway in the stated range, we may then compute the probability that the actual future population will fall within the limts specified by the consultant. This probability is given in column 3, and from the low values it is evident that such "high" and "low" projections are not truly indicative of actual variability. This is not unanticipated since they represent the particular deterministic projections that happened to yield the highest and lowest estimates respectively.

At the time of writing, preliminary census figures are available for some of the towns in the study region. Table 18 shows a comparison of the 1970 Model B projections and the LPVRPC planning projections with the preliminary Census figures. The figures in

POPULATION PROJECTION LONGMEADOW

YEAR	ACTUAL POPULATION (PLOT CODE 1)	ACTUAL POP.DENS. (SQ.MI.)
1940	5790	602
1945	6411	666
1950	6508	677
1955	8482	882
1960	10565	1098
1965	13809	1435

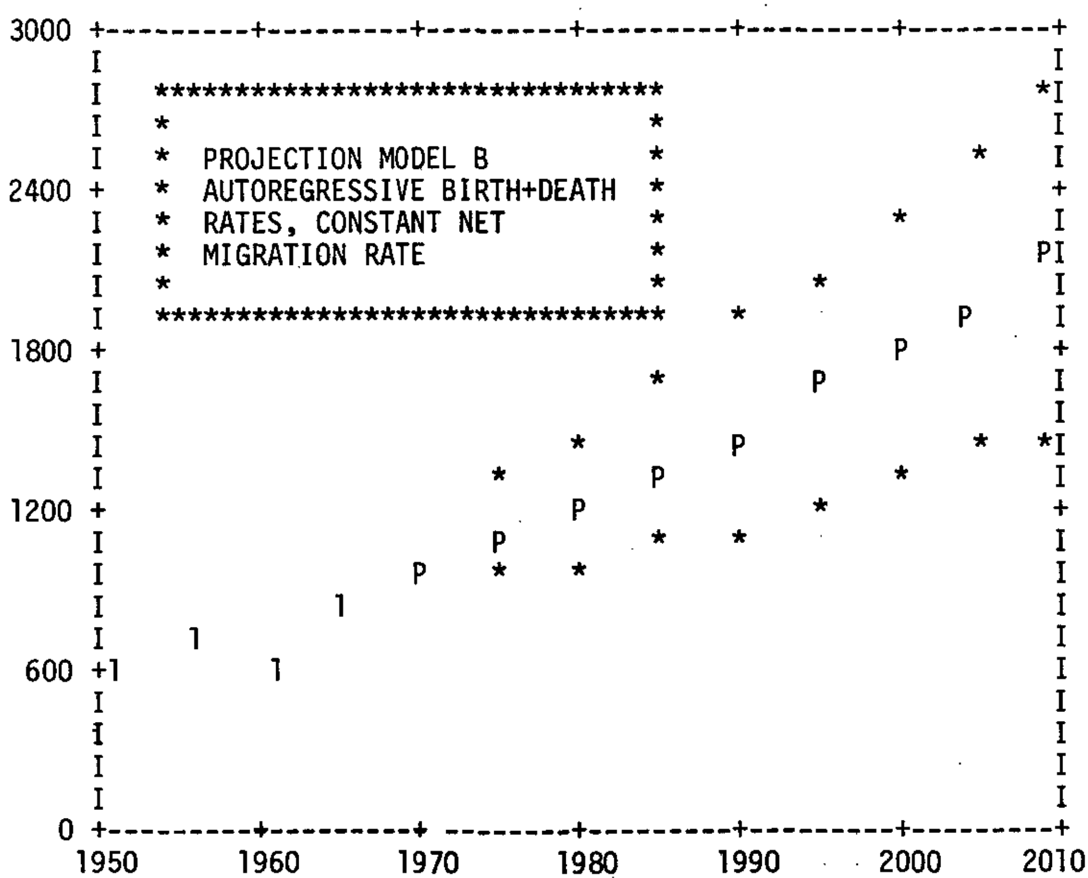
YEAR	HIGH PROJECTION	LOW PROJECTION	EXPECTED PROJECTION PERMANENT	EXPECTED PROJECTION SEASONAL	POP DENSITY
1970	17420	15040	16240		1688
1975	20940	17160	19040		1981
1980	24260	20160	22200		2309
1985	28460	22700	25580		2660
1990	32860	26280	29560		3075
1995	39620	30120	34860		3626
2000	46700	35140	40920		4254
2005	57220	40000	48620		5054
2010	66140	45700	55920		5814
code	(*)	(*)	(P)		

```
   69000 +---------+---------+---------+---------+---------+---------+
         I                                                          *I
         I   *******************************                         I
         I   *                             *                         I
         I   *  PROJECTION MODEL B         *                    *    I
   55800 +   *  AUTOREGRESSIVE BIRTH+DEATH *                        PI
         I   *  RATES, CONSTANT NET         *                         I
         I   *  MIGRATION RATE, N = 50     *                         I
         I   *                             *               *    P    I
         I   *******************************                        *I
   42600 +                                                           I
         I                                            *    P    *    I
         I                                                           I
POP      I                                            P    *         I
         I                                       *                   I
   29400 I                                  *    P    *              I
         I                                  P    *                   I
         I                             *                             I
         I                        *    P    *                        I
         I                        P    *                             I
   16200 +                   P    *                                  I
         I              1                                            I
         I         1                                                 I
         I    1                                                      I
         I1                                                          I
    3000 +---------+---------+---------+---------+---------+---------+
       1950      1960      1970      1980      1990      2000      2010
```

POPULATION PROJECTION BLANDFORD

YEAR			ACTUAL POPULATION (PLOT CODE 1)	ACTUAL POP,DENS. (SQ.MI.)
1940			479	9
1945			521	10
1950			597	11
1955			705	13
1960			636	12
1965			859	16

	HIGH PROJECTION	LOW PROJECTION	EXPECTED PROJECTION PERMANENT	EXPECTED PROJECTION SEASONAL	POP DENSITY
1970	1060	880	960		18
1975	1260	920	1080		21
1980	1400	980	1200		22
1985	1640	1020	1340		25
1990	1860	1060	1460		27
1995	2080	1200	1640		31
2000	2220	1280	1760		33
2005	2500	1420	1960		37
2010	2780	1460	2120		40
CODE	(*)	(*)	(P)		



$$\lambda_i^j = \frac{2}{5} \frac{\ell_i^j(5)}{(w_i^j + w_i^{j+5})} \quad \ldots\ldots\ldots\ldots\ldots \quad [\ 7.13\ ]$$

where w_i^j = population of region i in census year j

λ_i^j = average annual net migration rate over the interval j, j+5

Suppose that p consecutive estimates of net migration rates are available. Then the mean anual rate over the (5 x p)-year interval is given by

$$\bar{\lambda}_i = \frac{1}{p} \sum_{j=1}^{p} \lambda_i^j \quad \ldots\ldots\ldots\ldots\ldots \quad [\ 7.14\ ]$$

with standard deviation

$$s_i(\lambda) = \frac{1}{p} \left(\sum_{j=1}^{p} \lambda_i^{j(2)} - p\, \bar{\lambda}_i^{(2)} \right) \quad \ldots\ldots\ldots \quad [\ 7.15\ ]$$

and thus we may generate a sequence of time-homogenous migration rates[5]

$$\lambda_i^t = \bar{\lambda}_i + v^t s_i(\lambda) \quad \ldots\ldots\ldots\ldots\ldots \quad [\ 7.16\ ]$$

where v^t = RND(0,1)

Model A (see Table 16) is thus altered and augmented by Equations:

$$\lambda_i^t = \bar{\lambda}_i + v^t s_i(\lambda)$$

$$v^t = \text{RND}(0,1)$$

$$x_i^{t+1} = x_i^t (1 + \beta_i^t - \delta_i^t + \lambda_i^t)$$

	Model B1 (1)	Consultant (2)	(3)
Blandford	420	70	0.25
Pelham	650	90	0.23
Belchertown	2220	630	0.41
Longmeadow	4100	1700	0.55
Ludlow	9760	2000	0.31

Table 17 : Comparison of 1980 projection ranges
(see text for explanation of columns)

parentheses represent the difference to the census figure expressed as a percentage of this figure.

We note that the stochastic figure is nearer the census result than the consultant's estimate in 12 of 16 cases, a sufficient indication of superior prediction performance. In the 4 cases that the consultant was nearer (Pelham, Monson, E.Longmeadow and Ludlow), the differences are much less than for those where the stochastic method proved closer (see columns 2 and 4 of Table 18).

The worst stochastic result is for Ludlow, for which if the preliminary census figure is correct, the population has begun to decline. The stochastic model will not predict such a turning point for the expected value in its present form.

Of particular importance are the regional totals. For the 16 towns considered, the planning consultant is 7.2% high on the regional total, the stochastic model but 2% high. If the worst result is subtracted from each total (Ludlow for the stochastic, Springfield for the consultant's deterministic), these figures reduce to 1.5% and 5.5% for the stochastic and deterministic totals, respectively.

It is evident that if a method gives consistently high results, then nothing is gained by increasing the size of the projection region. A major advantage of the stochastic model is that it appears less biased than the traditional methods, with a better chance that errors in particular local area projections will cancel out for a regional total. This is of key importance for specifying the design capacity of regional waste treatment and water supply facilities.

	MODEL B (1)	Δ (2)	CENSUS (3)	Δ (4)	LPVRPD (5)
CHICOPEE	63680	-2820 (-4.2%)	66500	+3550 (+5.3%)	70050
SPRINGFIELD	168380	+6300 (+3.9%)	162078	+17170 (+10.5%)	179250
HOLYOKE	52300	+2760 (+5.6%)	49434	+5100 (+10.3%)	54540
WESTFIELD	30240	-860 (-2.8%)	31102	-1130 (-3.6%)	29970
AMHERST	25400	-770 (-2.9%)	26166	+4900 (+18.8%)	31050
NORTHAMPTON	27820	+100 (+0.3%)	27726	+2960 (+10.6%)	30690
HADLEY	3860	+127 (+3.2%)	3733	+817 (+22.0%)	4550
PELHAM	1000	+64 (+6.8%)	934	-44 (-4.7%)	890
PALMER	11510	-71 (-0.6%)	11581	-1020 (-9.7%)	10560
WARE	7910	-101 (-1.2%)	8111	-600 (-7.4%)	7511
MONSON	7670	+356 (+4.8%)	7314	-194 (-2.7%)	7120
E.LONGMEADOW	14840	+1831 (+13.0)	13009	+1141 (+8.0%)	14150
LONGMEADOW	16120	+280 (+1.8%)	15841	-1291 (-8.9%)	14550
LUDLOW	18620	+2854 (+18.0%)	15766	+1884 (+10.7%)	17650
SOUTH HADLEY	16680	-314 (-1.8%)	16994	+336 (+1.9%)	17330
W.SPRINGFIELD	28300	+45 (+0.2%)	28276	+1254 (+4.5%)	29530

Table 18 : Comparison of Model B and Planning Consultant projections with preliminary 1970 census results.

	MODEL B (1)	(2)	CENSUS (3)	(4)	LPVRPD (5)
CHICOPEE	75760 - 54600	YES	66500	YES	65000 - 71900
SPRINGFIELD	180860 - 155900	YES	162078	NO	187300 - 178500
HOLYOKE	54740 - 49860	NO	49434	NO	50000 - 50000
WESTFIELD	32400 - 28060	YES	30240	YES	30300 - 28500
AMHERST	30040 - 20860	YES	26166	NO	24700 - 24000
NORTHAMPTON	30780 - 24860	YES	27726	NO	29900 - 31500
HADLEY	4240 - 3480	YES	3733	NO	3300 - 3100
PELHAM	1140 - 880	YES	934	NO	880 - 820
PALMER	12600 - 10600	YES	11581		
WARE	8220 - 7610	YES	8111		
MONSON	8120 - 7220	YES	7314		
E. LONGMEADOW	17240 - 12460	YES	13009	NO	13500 - 14500
LUDLOW	20400 - 16820	NO	15766	YES	16600 - 15600
LONGMEADOW	18200 - 13960	YES	15841	NO	13000 - 12300
SOUTH HADLEY	20240 - 13100	YES	16994	NO	17800 - 18800
W. SPRINGFIELD	30140 - 26500	YES	28276	YES	29500 - 27900

Table 19 : Comparison of stochastic and deterministic projection ranges for 1970 with preliminary census results.

For example, the Engineering consultant to the LPVRPD has recommended that the feasibility of regional waste treatment facilities be examined for Amherst, Hadley and Pelham. For this region the stochastic projection is -1.9% low (for the 1970 projection), the planning consultant 18.5% high. It appears unlikely that such discrepancies will not also exist for longer range projections. In recent weeks several newspaper articles have also discussed the possibility of regional waste treatment facilities in the South Hadley - Holyoke area. We note that the planning consultant's deterministic projection for both these towns together is high by 5436 for the 1970 projection, the stochastic projection only 2446 high, again underscoring the need for an unbiased projection method.

Table 19 compares the projection ranges for the same set of towns.[6] No range was indicated by the LPVRPC for Palmer, Ware and Monson, since these towns joined the LPVRPD only last year. The census figure falls within 14 of 16 stochastic ranges, but within only 4 of 13 deterministic ranges as projected by the planning consultant.

It may be concluded that the stochastic model yields demonstrably superior results for short-term local area projections. A shortcoming of the model in its present form is its inability to predict turning points: further developments will hopefully rectify this deficiency.

---

[6]The ranges on Table 19 represent the population projections by a different consultant to the LPVRPD, and are therefore not always consistent with those of Table 18

The Interregional Projection Model

The projections of Model B assume constant mean net migration rates into the future. It is clear, however, that many communities will enter declining rate growth phases as the most advantageous residential sites become developed. A necessary refinement to extend the short-term projections is to formulate this time-dependence of growth and migration rates in terms of explanatory variables that are more readily projected than the growth rates themselves, using for example the equations of Table 15. Unfortunately the coefficients of these equations can be obtained only from a cross-section analysis, as many of the explanatory variables (especially land-use data and exact information as to the extent of service areas) are available only for isolated years. The values of the explanatory variables themselves, however, can be updated as an integral part of the projection process with relative ease.

A more serious limitation to the constant rate projection Model is the implied abstraction from the regional entity. The extent to which a community can expect to grow will depend not only on its relative locational and service advantages but also on the overall regional in- and out-migration rates that are dependent on employment opportunities and the development of the economic base rather than opportunities for residential expansion.

The interregional model will be applied to the 13 suburban towns of the LPVRPD, as defined on Figure 13. The flow chart of the

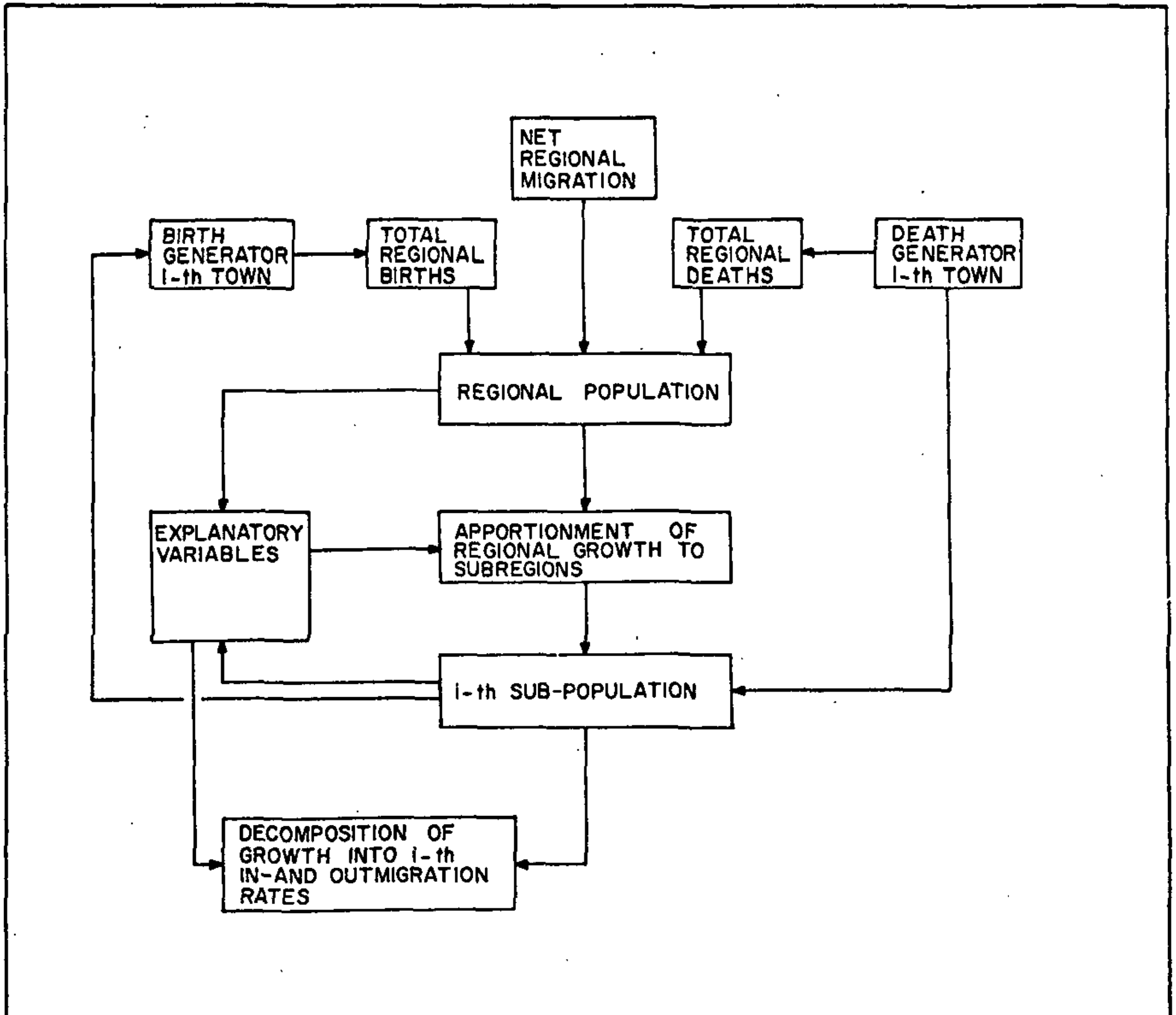


Figure 26 : Flow Chart, interregional projection model C (PROGRAM POPC)

computerized model is shown on Figure 26 and should be referenced during the explanation of individual steps in the following sections.

Regional growth rate. Regional births and deaths are obtained by summation of births and deaths for the constituent towns, which are generated as in Models A and B by the first order autoregressive equations of Table 16. Total regional net migrations are generated as in Model B by Eq.[7.16] from which one thus obtains total regional population at each projection step. The equations of the sector are:

$$b_T^t = \sum_{i=1}^{k} b_i^t \qquad d_T^t = \sum_{i=1}^{k} d_i^t$$

$$\lambda_T^t = \bar{\lambda}_T + v^t s_T(\lambda) \quad \text{(Eq.[7.16])}$$

$$\ell_T^t = \lambda_T^t \, x_T^t$$

$$q_T = \ell_T^t + b_T^t - d_T^t \quad \text{. . . . . [7.17]}$$

$$\phi_T^t = q_T^t / x_T^t \quad \text{. . . . . [7.18]}$$

$$x_T^{t+1} = x_T^t + q_T^t \quad \text{. . . . . [7.18]}$$

where the subscript T indicates a figure for the total region, and where

q_i^t = growth increment for town i in period t

ϕ_i^t = growth rate of town i during period t

Decomposition of regional growth rate. The relationship between the growth rate of the i-th constituent community, ϕ_i^t relative to the overall regional growth rate ϕ_T^t is given by Eq.[6.11], namely

$$\phi_i^t = \phi_T^t \, a_o \, [\prod_{j=1}^{4} (X_{ji})^{(a_j)}] \, u \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 6.11\ ]$$

where $\quad a_i$ = empirically determined coefficients

$\quad u$ = random term distributed such that $\log_e u \sim N(0,\sigma\ )$

but since the sum of individual growth increments q_i^t must equal the total regional growth increment q_T^t ,

$$\sum_{i=1}^{k} q_i^t = q_T^t \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 7.20\ ]$$

the initial estimate of q_i^t from Eq.[6.11] is adjusted such that

$$\tilde{\phi}_i^t = \phi_i^t \frac{q_T^t}{\sum \phi_i^t \quad x_i^t} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 7.21\ ]$$

where $\tilde{\phi}_i^t$ is the corrected estimate of the growth rate. The explanatory variables X_{ji} are not constants, but will vary through time. Hence

$$\phi_i^t = \phi_T^t \, a_o \, [\prod_{j=1}^{4} (X_{ji}^t)^{(a_j)}] \, u \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 7.22\ ]$$

Explanatory variables. From the aforegoing elaborations it follows that at each time point of the projection the explanatory variables require updating. It should again be noted that the equations discussed in this section are specific to the LPVRPD, and require re-evaluation on availability of the 1970 census results or on application of the interregional model to other planning regions. The 4 variables that determine growth rates (see Eq.[6.11]) are : Fraction sewered (FRACSW), ratio of single family housing to total residential area (SFHR), vacant acres (VACAC) and residential density (RESDEN). Introducing new variables (dropping the i subscripts for clarity), and the FORTRAN identification as utilized in the computer program:

RESAC(J) = total residential area, acres, period J

SFAC(J) = single family housing, acres, period J

DEVA(J) = acres developed for residential location during period J

XPOP(J) = population at end of period J

GRV(J) = growth increment, period J

GRR(J) = growth rate, period J

F1(J) = fraction of total acres developed accruing to to single family housing, period J

F2(J) = average lot size, period J

PPH(J) = average persons per household, period J

New values of the explanatory variables are then obtained at each time-period by application of the following equations:

```
GRV(J+1) = f(FRACSEW(J),SFHR(J),VACAC(J),RESDEN(J))
              (equation [ 6.11 ])

GRR(J+1) = GRV(J+1)*XPOP(J)

DEVA(J+1)= (GRR(J+1)/PPH(J))*F2(J+1)

VACAC(J+1)= VACAC(J) - DEVA(J+1)

RESAC(J+1)= RESAC(J) + DEVA(J+1)

RESDEN(J+1) = (XPOP(J)+GRR(J+1))/RESAC(J+1)

SFAC(J+1) = SF(J) + DEVA(J+1)*F1(J+1)

SFHR(J+1) = SFAC(J+1)/RESAC(J+1)
```

Values of these variables for the total region required in Eq.[6.11] are obtained from appropriate weighted summation. A discussion of the variable FRACSEW is deferred to Chapter IX. The variables F1 and F2 are exogenous and must be specified by the decision-maker for the terminal year; intermediate values are obtained by some interpolation rule. The sensitivity of the projections to the specification of these exogenous variables is examined below.

Results. Table 20 compares the Model C projections for the LPVRPD suburban ring towns with the planning consultant projections. The regional net migration generator was adjusted for this set of projections such that the rate for the year 2000 was zero, intermediate values interpolated linearly. No particular significance can be attached to the significant discrepancies between the two sets of projections in view of the deficient data-base upon which the coefficients of Eq. [6.11] were estimated. Preliminary census results are available for the following towns: Westfield 31102 (v. 30840), West Springfield 28276 (v. 28720), Longmeadow 15841 (v. 15520), East Longmeadow 13009 (v. 13480), South Hadley 16994 (v. 15600), Ludlow 15766 (v. 18860), where the figure in parentheses represent the Model C projection. A full evaluation must await more complete census data. Although there is some discrepancy for South Hadley and Ludlow, we note that for the six towns together, Model C is 2000 high (= 1.6 percent). Although there can be no immediate judgment on the accuracy of the longer-term stochastic projections, it is reasonable to suppose that the deterministic projections will not be more accurate for longer-term forecasts in view of their demonstrated inferiority for short-term predictions.

Figure 27

POPULATION PROJECTION LONGMEADOW

YEAR	ACTUAL POPULATION (PLOT CODE 1)	ACTUAL POP.DENS. (SQ.MI.)
1940	5790	602
1945	6411	666
1950	6508	677
1955b	8482	882
1960	10565	1098
1965	13809	1435

	HIGH PROJECTION	LOW PROJECTION	EXPECTED PROJECTION PERMANENT	EXPECTED PROJECTION SEASONAL	POP DENSITY
1970	12260	4880	15560		
1975	18780	15940	17360		
1980	21680	16780	19240		
1985	24100	18080	21080		
1990	26280	19140	22700		
1995	27780	20480	24120		
2000	29160	21680	25420		
CODE	(*)	(*)	(P)		

```
30000 +---------+---------+---------+---------+---------+---------+
      I                                                          *I
      I  *********************************                   *    I
      I  *                               *                        I
      I  *  PROJECTION MODEL C           *              *         I
25000 +  *  INTERREGIONAL MODEL OF LPVRPC*         *             P+
      I  *  SUBURBAN RING.               *                   P    I
      I  *                               *              P         I
      I  *                               *    *                  *I
      I  *********************************         P              I
20000 +                                                      *    +
      I                                  *    P         *         I
      I                                            *              I
      I                                  P    *                   I
      I                             P*   *                        I
15000 +                             *                             +
      I                        1                                  I
      I                                                           I
      I                                                           I
      I                   1                                       I
10000 +                                                           I
      I                                                           I
      I              1                                            I
      I         1                                                 I
      I1    1                                                     I
 5000 +---------+---------+---------+---------+---------+---------+
    1940      1950      1960      1970      1980      1990      2000
```

	LPVRPC		MODEL C	
	1980	1990	1980	1990
AGAWAM	26000	31800	25500	29900
EASTHAMPTON	14600	15200	15400	19280
EASTLONGMEADOW	18750	25100	16200	19300
GRANBY	8600	10500	5980	7380
HAMPDEN	4590	6000	3820	4700
LONGMEADOW	16320	19500	19240	22700
LUDLOW	21300	24100	24540	28420
SOUTHAMPTON	3610	5000	2900	3500
SOUTH HADLEY	22510	28210	18480	22480
SOUTHWICK	9850	13100	7000	8740
WESTFIELD	34200	38000	40980	47540
WILBRAHAM	13760	16300	13900	16400
WEST SPRINGFIELD	34580	38000	37500	43640

Table 20 : Model C projections for 1980 and 1990

Since the prior determination of terminal regional net migration, average lot size and relative proportion of single family housing represent judgments of future conditions, it is prudent to include some measure of uncertainty in their specification. Thus, for example, the average lot size in the terminal year N is specified as a fraction of the present lot size (f_1) plus or minus some quantification of the estimated reliability of this judgment. The program assumes that

$$Fl(N) = Fl(1)*f_1 + v^t\sigma(f_1)$$

$$v^t = RND(o,1)$$

$$\sigma(f_1) = \text{subjective estimate of specification variability}$$

In Chapter IX we shall suggest an improved method of estimating this particular exogenous variable.

Table 21 illustrates the effect of changes in the specification of exogenous variables on the 1980 and 1990 projections for East Longmeadow. Case 1 assumes constant birth and death rates and zero regional net migration for the total region by 2000. Cases 2 and 3 demonstrate the insensitivity of the projection to decreased birth rates. Case 4 assumes constant rather than decreasing regional net migration, and Case 5 assumes that the average lot size of future single family homes will increase by 20 percent.

By implementing such an interregional projection model on time-sharing, the planner has at his disposal a tool that permits immediate quantitative assessment of the potential effectiveness of policy decisions (for example, increasing minimum lot size in a zoning

	1980	1990
Case 1 (see text)	16200	19300
Case 2 (Mean birthrate $\bar{\beta}$ declining by 20% by the year 2000, for all 13 towns in regional set)	16000	18780
Case 3 (Mean birthrate for East Longmeadow down 20% by 2000, other towns constant)	16120	19120
Case 4 (Constant net regional migration)	16600	25500
Case 5 (as Case 1 but with average lot size up 20 % by 2000)	16000	18920

Table 21 : Model C projections for East Longmeadow

regulation) or the sensitivity of the model to professional judgments (such as future birth and death rates). This type of simulation model has long been available to corporate managements as decision-making aid, and it may be reasoned that if planning and engineering consultants were also to be held accountable for the accuracy of their forecasts upon which capital investments are based, population projection would cease to be a process of artistic guesswork to become a scientific method, utilizing modern quantitative techniques.

CHAPTER VIII

A MODEL OF URBAN GROWTH

Spatial variations of urban population densities. In a classic paper, Clark (81) postulated that urban population densities decline exponentially with distance from the city centre. This relationship may be formulated as

$$h_r = h_o e^{-br} \qquad [8.1]$$

where h_r = population density at distance r from the city centre

h_o = population density at the city centre

b = density gradient

or, in its frequently utilized logarithmic form

$$\log_e h_r = \log_e h_o - br \qquad [8.2]$$

which is linear in semi-logarithmic coordinates. Subsequently, numerous studies have confirmed Clark's findings in that a statistically significant fit to the negative exponential relationship has been found for all but oriental cities.[1] Indeed, Berry (87) states

[1]See Berry et al. (82) for a comprehensive review of the literature to 1963. More recent contributions include Newling (83), Wilkins (84), Wilkins and Shaw (85) and Treadway (86).

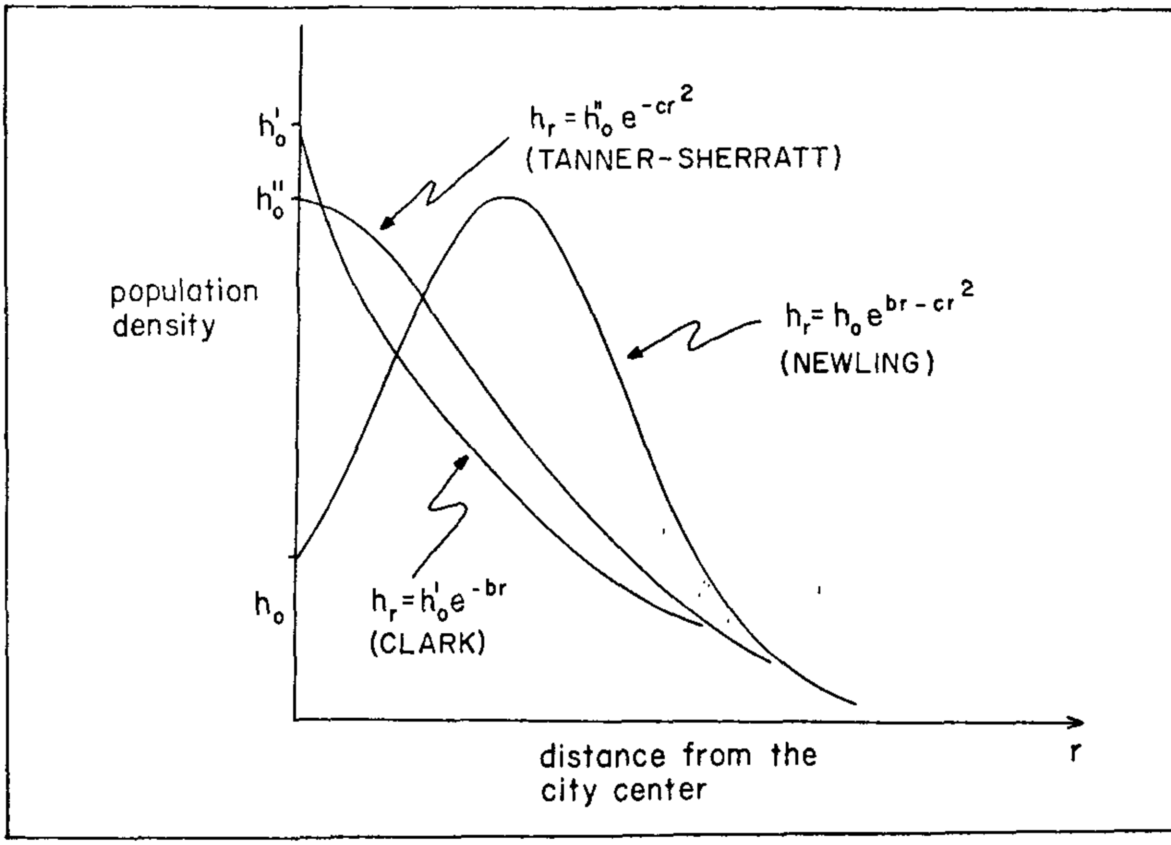


Figure 28 : Alternative formulations of urban distance-density radial profiles.

" ... the negative exponential decline of phenomena with increasing distance from the city center is nowhere more apparent than in urban population densities. Regardless of time or place, this is the pattern to be found: in some four hundred cases examined so far there have been no exceptions. Here, therefore, is a finding of great generality."

Muth (88) examined forty-six U.S. cities for the year 1950 and found the variability in the density gradient to be strongly dependent on transportation costs.[2] Muth also demonstrated that if certain simple assumptions were made about the price-distance, demand and production functions,[3] the decline of population density would indeed be negative exponential. Tanner (89) and Sherratt (90) proposed independently a revision of Clark's model such that urban population densities decline exponentially with the square of distance;

$$h_r = h_o e^{-cr^2} \quad \ldots\ldots\ldots\ldots\ldots \quad [\ 8.3\ ]$$

Newling (91) suggested a quadratic exponential formulation that recognizes the emergence of a density crater in the central business district, namely

$$h_r = h_o e^{br - cr^2} \quad \ldots\ldots\ldots\ldots\ldots \quad [\ 8.4\ ]$$

Figure 28 shows that both the Newling and Tanner-Sherratt formulations describe bell-curved density profiles and that their logarithmic

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[2]As measured by the surrogates miles of local transit system per unit area and vehicle miles operated per mile of line.

[3]Log linearity of demand and production functions, negative exponential price distance relationship.

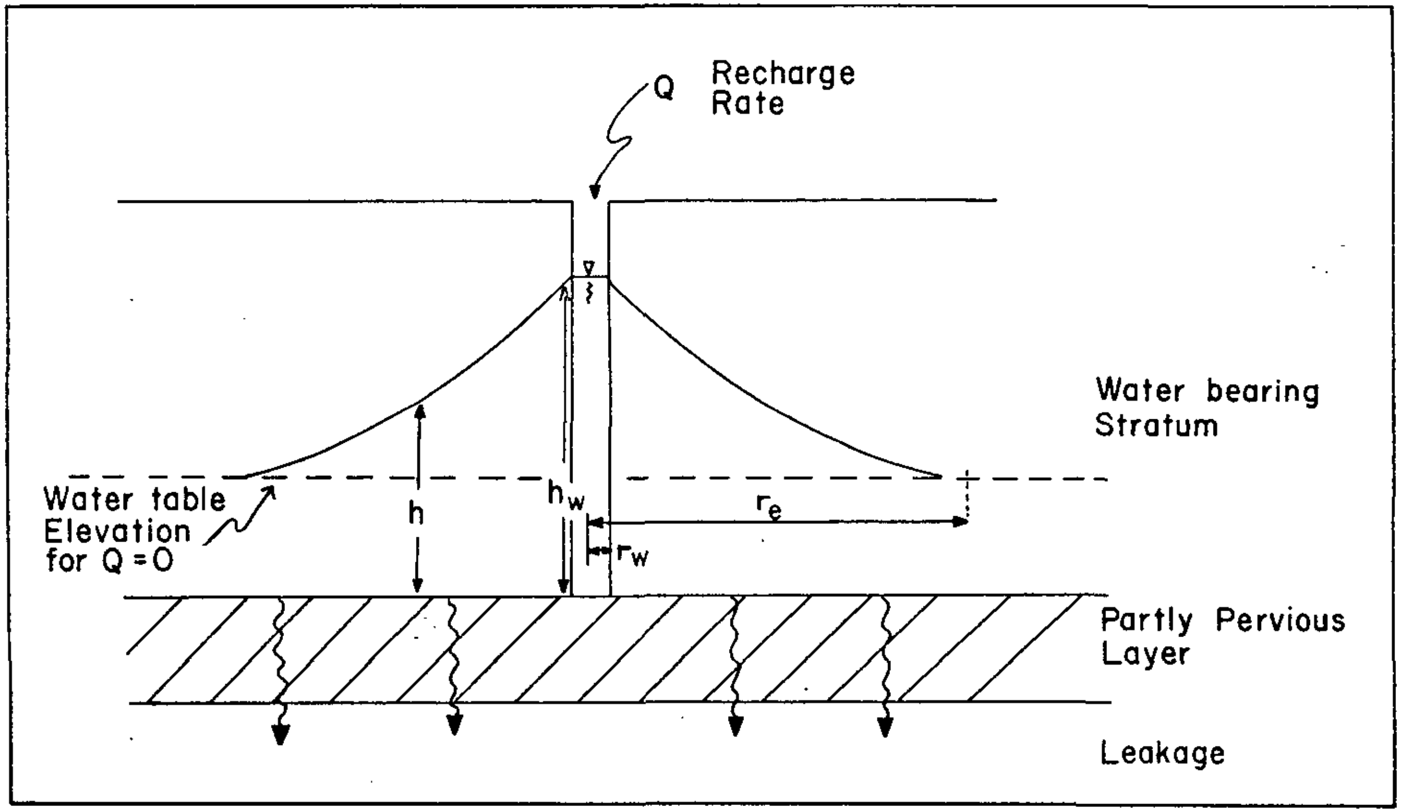

Figure 29 : The recharge well - definition sketch

transformations are therefore parabolic, concave downward. Newling suggests that

> " ... this downward concavity (of the log transformed density profile) approximates more closely the observed density profiles of cities than does the linear curve generated by the logarithmic transformation of Clark's negative exponential model." (Newling, op. cit., p. 243)

It is quite evident that several functional forms will provide an adequate statistical fit to observed density profiles. We note, however, that the Tanner-Sherratt formulation, developed on empirical grounds, describes the analytical solution to a simple heat diffusion problem. It is therefore suggested that the process of urban growth itself can be visualised as a diffusion problem. The particular density-distance relationship that results is then dependent on the fundamental assumptions of the diffusion problem, rather than on empirical criteria of "best-fit".

A physical analogy that appears singularly appropriate is that of the recharge well (see Figure 29). The solutions that describe spatial variations in piezometric head in the vicinity of this recharge well correspond closely to the distance-density relations of an urbanized area. The differential equations that govern such a physical problem may be developed by consideration of an infinitesimal element of aquifer volume within the framework of fundamental physical laws; in the recharge well problem, the law of mass conservation and Darcy's Law are applicable. Analogous equations will be developed in the following section; in place of the aquifer element we shall consider an infinitesimal spatial element of geographic space.

The dynamic nature of urban systems implies changes in the constants of Equation [8.4] through time. As Winsborough (92) noted,

> " ... an increase in population size must be accommodated by an increase in congestion (or central density) or a decrease in concentration (density gradient), or some compensatory change in both."

By modification of Eq. [8.4] to include a time component, changing density profiles are generated with relative ease. For example, Newling's formulation

$$h_r^{\,t} = h_o^o \, e^{mt - nt^2 + (b_o + gt)r - cr^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad [\ 8.5\ ]$$

where $h_r^{\,t}$ is the density at distance r at time, and m, n, b_o, g, c are constants, generates a sequence of curves describing the emergence of a density crater in the CBD with increasing time (Newling, loc. cit., p. 249). Again, however, this device is an empirical artifact. A logically more consistent approach is to view the dynamic nature of an urban system in terms of the demographic events that sustain it. For the groundwater well, it is clear that once recharge stops, the water level in the unconfined aquifer will descend to the original water table level. In analogy, therefore, the role of in-migration in sustaining a density gradient becomes apparent. Out-migrations to places outside the region under consideration can be visualized as leakage through a partly permeable stratum at the base of the aquifer. Migrations within the sphere of influence of the well occur in the horizontal plane. The solution techniques of partial differential equations are sufficiently developed for the dynamic nature of the problem to be directly incorporated in the diffusion analogy. Although the unhomogeneous nature of geographic space may preclude explicit analytical solution of the resulting differential equations, application of finite difference techniques permits rapid numerical solution.

Plant and animal ecologists have long utilized diffusion models to describe the spatial distribution of particular species whereby the differential form of the postulated growth law (geometric, logistic, etc.) is augmented by a second order diffusion term, conceptually the limiting form of a two-dimensional random walk.[4] Most of these models have been linear, for which solution techniques are routine. The differential equations that arise in the urban growth model, however, are non-linear, and we turn now to their derivation.

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[4]The classic reference is Skellam (93). Watt (94) reviews more recent work in quantitative ecology.

Derivation of the Urban Density Equation. Consider an infinitesimal spatial element of dimensions δx, δy (see Figure 30), each having the dimension of length [L]. Let the residential density of this element be denoted h, with dimensions of persons per unit area $[M]/[L]^2$. The principle of mass conservation for this element requires that the mass inflow rate equals the sum of the mass outflow rate and the rate of change of mass storage within this element.

For a given geographical spatial element we may identify four types of demographic change: births, deaths, inmigrations and outmigrations. In the context of the physical analogy, such events represent changes in mass. Births and deaths may be visualized as mass inflow and outflow along the z-axis. Inmigration from adjacent elements and outmigrations to adjacent elements constitute mass in- and outflows in the horizontal x-y plane. Such migratory movements, representing changes in residential location within the region under consideration (intra-regional migrations) are to be differentiated from migrations to and from other regions outside the immediate sphere of influence (inter-regional migration). It is hypothesized that all inmigrants from outside enter the region at the center, whereas inter-regional outmigrants depart from all locations. The latter may thus be visualized as mass outflow in the z direction. The birth, death and interregional outmigration rates will be denoted as β, δ, ω respectively.

Consider first the components of the migration velocities in the x-y plane. Let the component of the migration velocity in the x-direction at the center of the element be denoted $u(x_o, y_o)$.

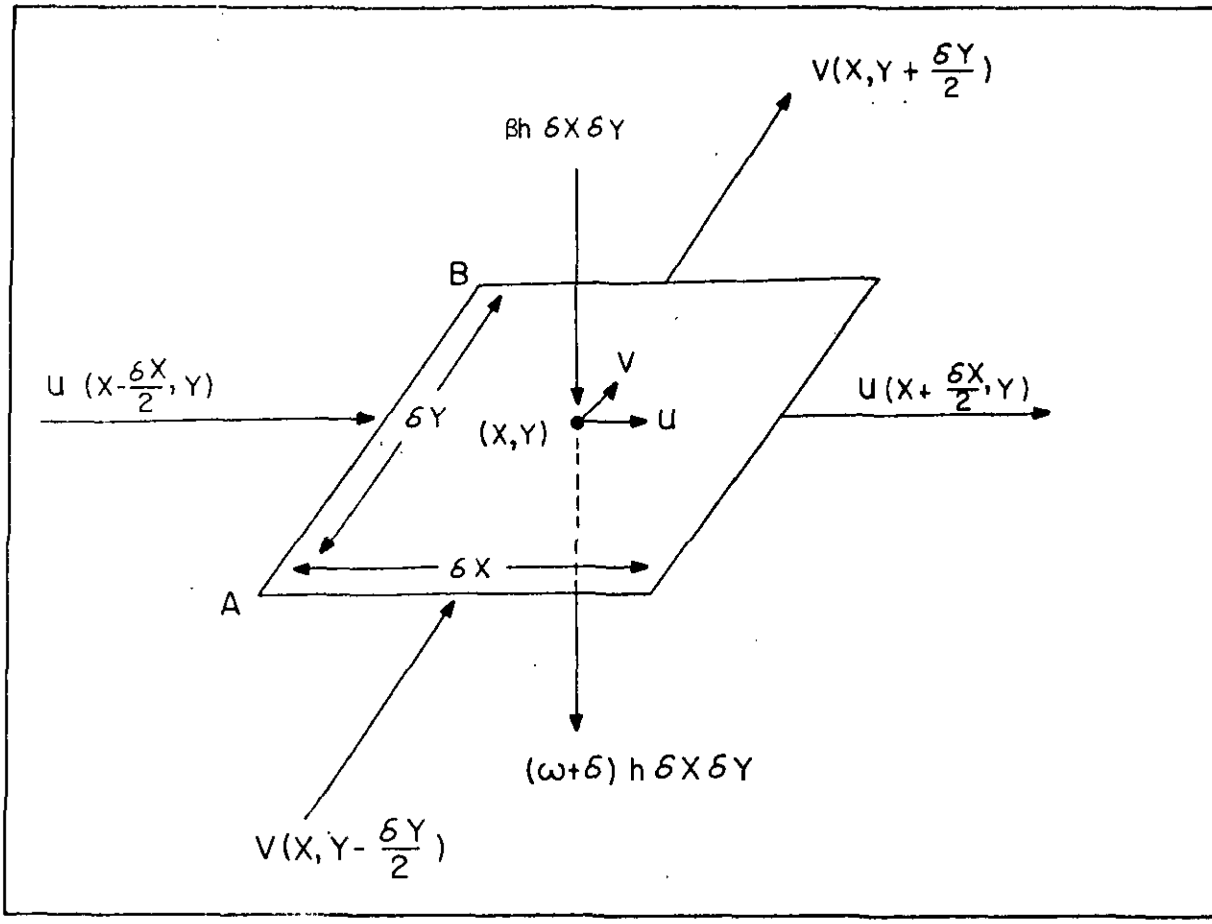

Figure 30 : The infinitesimal spatial element, cartesian coordinates

Then the corresponding population mass flux is [5]

$$u(x_o,y_o)\ \frac{[L]}{[T]}\ \delta y\ [L]\ h\ \frac{[M]}{[L]^2} = \frac{[M]}{[T]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad [\ 8.6\ ]$$

where [L], [T], and [M] are the corresponding dimensional units. Population is thus represented by the dimension mass [M]. To find the value of u through the edge AB at $x_o \pm \delta x/2$ we may use the Taylor series expansion

$$u(x_o \pm \frac{\delta x}{2}, y_o) = u(x_o,y_o) \pm \frac{\delta x}{2}\frac{\partial u(x_o,y_o)}{\partial x} \pm \frac{1}{2}(\frac{\delta x}{2})^2 \frac{\partial^2 u(x_o,y_o)}{\partial x^2}$$

$$\pm \ . \ . \ . \ \pm \quad . \quad . \quad . \qquad [\ 8.7\ ]$$

Neglecting higher order terms, the population flux through the AB edge is approximated by

$$[u\ h - \frac{\delta x}{2}\frac{\partial (uh)}{\partial x}]\ \delta y$$

and we may write for the mass conservation equation

$$\underbrace{[u\ h - \frac{\delta x}{2}\frac{\partial (uh)}{\partial x}]\ \delta y}_{\substack{\text{inmigration} \\ \text{x-direction}}} + \underbrace{[v\ h - \frac{\delta y}{2}\frac{\partial (vh)}{\partial y}]\ \delta x}_{\substack{\text{inmigration} \\ \text{y-direction}}} + \underbrace{\beta\ h\ \delta x\ \delta y}_{\text{births}}$$

$$- \underbrace{[u\ h - \frac{\delta x}{2}\frac{\partial (uh)}{\partial x}]\ \delta y}_{\substack{\text{outmigration} \\ \text{x-direction}}} + \underbrace{[v\ h - \frac{\delta y}{2}\frac{\partial (vh)}{\partial y}]\ \delta x}_{\substack{\text{outmigration} \\ \text{y-direction}}} + \underbrace{(\delta+\omega)\ h\ \delta x\ \delta y}_{\substack{\text{deaths} \\ \text{interregional} \\ \text{outmigrants}}}$$

---

[5]The corresponding term for the unconfined groundwater flow problem, that considers the mass flux in an infinitesimal element of dimensions $\delta x \times \delta y \times \eta$ is given by

$$u(x_o,y_o)\frac{[L]}{[T]}\ \rho\ \frac{[M]}{[L]^3}\ \delta y\ [L]\ \eta\ [L] = \frac{[M]}{[T]}$$

where ρ is the fluid density and η the z-axis dimension of the infinitesimal element (=free surface height)

$$= \frac{\partial h}{\partial t} \delta x \; \delta y \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad [\ 8.8\ ]$$

change of mass storage in time

Collecting terms,

$$- \frac{\partial (uh)}{\partial x} \delta x \; \delta y - \frac{\partial (vh)}{\partial y} \delta x \; \delta y \; + (\beta - \delta - \omega) \; \delta x \; \delta y \; \; h = \frac{\partial h}{\partial t} \delta x \; \delta y$$

Expressing the migration velocity components u,v in terms of the Darcy Law analogy,

$$u = - K_x \frac{dh}{dx} \qquad v = - K_y \frac{dh}{dy} \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad [\ 8.10\ ]$$

and substituting these expressions in Eq.[8.9]

$$\frac{\partial}{\partial x} \left(K_x \frac{\partial h}{\partial x} h \right) + \frac{\partial}{\partial y} \left(K_y \frac{\partial h}{\partial y} h \right) + (\beta - \delta - \omega) \; h \; = \; \frac{\partial h}{\partial t} \; [\ 8.11\ ]$$

But since K_x, K_y are constants,

$$\frac{K_x}{2} \frac{\partial^2 h^2}{\partial x^2} + \frac{K_y}{2} \frac{\partial^2 h^2}{\partial y^2} + (\beta - \delta - \omega) \; h \; = \; \frac{\partial h}{\partial t} \; . \; . \; . \; . \qquad [\ 8.12\ ]$$

Development of applicable boundary conditions is facilitated by examination of the radially symmetric case for which the dimensions of the problem are reduced.

In radial coordinates (θ, r) we again consider the conservation of mass in the infinitesimal element (see Figure 37), which may be written as

$$[v_r\, h - \frac{\partial (h v_r)}{\partial r} \frac{\Delta r}{2}]\ (r - \frac{\Delta r}{2})\ \Delta\theta + h\ \beta\ \Delta\theta\ \Delta r\ r$$

mass inflow rate

$$-[v_r\, h + \frac{\partial (h v_r)}{\partial r} \frac{\Delta r}{2}](r + \frac{\Delta r}{2})\ \Delta\theta - h\ (\omega + \delta)\ r\ \Delta\theta\ \Delta r$$

-mass outflow rate

$$= \frac{\partial h}{\partial t}\ r\ \Delta r\ \Delta\theta \qquad [\ 8.13\ ]$$

= change of mass storage in time

Collecting terms, and again introducing the Darcy Law analogy for v_r, the velocity component in the r-direction (for radial symmetry $v_\theta = 0$, $\Delta\theta = 2\pi$)

$$\frac{\partial}{\partial r}\ (h\ K\ \frac{\partial h}{\partial r}) + K\ \frac{\partial h}{\partial r}\ h + h\ (\beta - \delta - \omega)\ r = \frac{\partial h}{\partial t}\ r \qquad [\ 8.14\ ]$$

hence

$$r\ \frac{K}{2}\ [\frac{\partial^2 h^2}{\partial r^2} + \frac{1}{r}\ \frac{\partial h^2}{\partial r^2}] + h\ (\beta - \delta - \omega)\ r = \frac{\partial h}{\partial r}\ r \qquad [\ 8.15\ ]$$

and, finally

$$\frac{\partial^2 h^2}{\partial r^2} + \frac{1}{r}\ \frac{\partial h^2}{\partial r} + \frac{2}{}\ h\ (\beta - \delta - \omega) = \frac{2}{K}\ \frac{\partial h}{\partial t} \qquad [\ 8.16\ ]$$

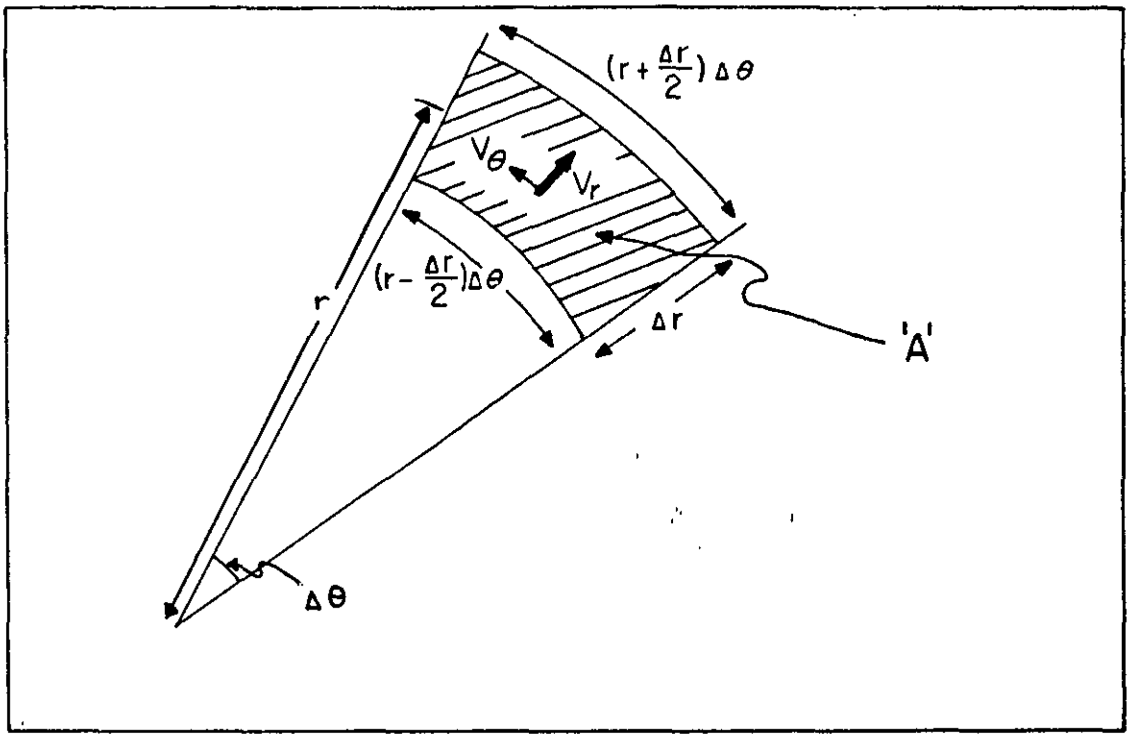

Figure 31 : The infinitesimal spatial element, radial coordinates

This equation is again analogous to that obtained for radially symmetric unconfined flow with the addition of the term in h.

The first boundary condition is $h = \xi$ for some $r = r_e$, where ξ is some residual rural population density beyond the urbanized area. The second boundary condition may be obtained by consideration of the conditions at the point of entry of the inter-regional inmigrants. As in previous sections, let μ be the in-migration rate, and let v_w be the centrifugal velocity across a circular perimeter at a distance r_w from the centre. Then provided no births or deaths occur within this perimeter,

$$v_w \, h \, 2 \, \pi \, r_w = \mu \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 8.17\ ]$$

Dimensionally

$$v_{r=r_w} \frac{[L]}{[T]} \quad h_{r=r_w} \frac{[M]}{[L]^2} \quad 2 \, \pi \, r_w \, [L] \; = \mu \, \frac{[M]}{[T]}$$

Expressing v in terms of the Darcy Law analogy

$$- K \frac{\partial h}{\partial r} \; h \; 2 \; \pi \; r_w \; = \mu \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 8.18\ ]$$

hence $$h \frac{\partial h}{\partial r} = \frac{-\mu}{2\pi r_w K}$$

Integrating
$$\int h \frac{\partial h}{\partial r} \, \partial r = \int \frac{-\mu}{2\pi r_w K} \, dr_w$$

$$\frac{1}{2} h^2 = - \frac{\mu}{2\pi K} \log_e r + C \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; [\ 8.19\ ]$$

Unfortunately, we cannot evaluate the constant C, since for $r > r_w$, the relation [8.17] no longer holds as a result of births, deaths and inter-regional outmigrations occurring within any perimeter greater than $2\pi r_w$. An approximation may be obtained by setting

$$\mu^* = \mu + (\beta - \omega - \delta) \sum_{ij} h_{ij}$$

and then for $r = r_e$, $h = \xi$

$$C = \frac{\xi^2}{2} + \frac{\mu^*}{2\pi K} \log_e r_e$$

and hence finally

$$h_w = \sqrt{\frac{1}{K\pi} (\mu^* \log_e r_e - \mu^* \log_e r_w) + \xi^2} \quad . \; . \; . \; [8.20]$$

which is the required second boundary condition. The analogy to the recharge well equates the well casing location to the limits of the Central business district (CBD). In actuality, delimitation of the CBD may involve some subjective judgements, and the literature of urban geography is not without controversy on this point (see e.g. Brill (95)). A direct evaluation of Eq.[8.18] by a finite difference approximation is given in the next section.

Implications of the model. Equation [8.13] describes spatial variations of population density in an urban system. In the derivation of this equation, an analogy to Darcy's Law was employed. Darcy's Law relates the flow velocity of a fluid in a porous medium to the head-loss, and quantifies the proportionality of velocity to the energy gradient and a constant known as the permeability (or hydraulic conductivity). In applying Darcy's Law to the urban growth situation, we postulate that intra-urban migration velocity is proportional to the gradient in population density. That is, the rate of movement from one residential location to another is governed by relative spatial differentials in density. The constant of proportionality (permeability), which indicates the extent to which the aquifer medium resists the passage of water, represents the propensity of a particular location to the passage of intra-urban migrants seeking new (or intensifying existing) residential location. An intact major physiographic barrier will inhibit residential location whatever the density gradient. The physical analogy may indeed be carried one step further. The permeability,[7] K, of a particular porous medium, may be decomposed into the product of the intrinsic permeability, K*, which is a characteristic of the medium itself, and the ratio of the fluid properties specific weight and dynamic viscosity; the latter being a function only of the fluid in question, i.e.

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[7]Hydraulic permeability has dimensions [L]/[T].

prevailing views of sociologists. Johnston (96) appears to summarize the concensus:

> " ... the majority of in-migrants to a city should first reside in the inner residential areas where their arrival will initiate a process of invasion and succession. Within the suburbs, the greater competition for housing in inner areas should encourage most movers to proceed to destinations closer to the urban periphery, so that the general migratory trend within the city should be away from the centre."
>
> " ... city expansion is largely a function of household formation, whether by in-migration from elsewhere or by persons already resident in the city either by marriage or leaving the parental home to live alone. Much of the physical expansion takes place through the construction of new units on the urban periphery. The first inhabitants of such new dwellings generally are not newly formed households. Most people who occupy these homes are already resident in the city, and the new households generally occupy second-hand dwellings. The result is a continuous and complex pattern of intra-urban migration."

A limitation is, of course, that patterns of radial invasion-succession have been most studied in the largest metropolitan areas where racial or ethnic implications have focused attention.[8] Nevertheless, our hypothesis is fundamentally in agreement with the Burgess Model.[9] The asymmetries of the real urban area that distort the ideal concentric zonal pattern envisaged by Burgess does not detract from the overall validity of the postulated driving force mechanism: indeed, in terms of the diffusion model, the inhomogeneity of permeability will in itself result in distortions of the ideal density profile.

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[8]See especially Tauber and Tauber (97) and Ford (98).

[9]Schnore (99) gives a lucid exposition of this classic urban model and its ramifications.

Empirical analysis of the growth and migration rates in the Springfield SMSA (see Chapter VI) shows that of all the variables considered, variations in population density were the most important single explanatory factor (variable X_9 of Table 14). Adams (100) lends further support to our hypothesis by his findings of strong directional bias in intra-urban migration. Defining the move angle as the angle subtended at the CBD between successive residential locations (see Figure 32), he found in one sample that the most probable move angle was less than 10 degrees, implying centrifugal movements to be dominant.

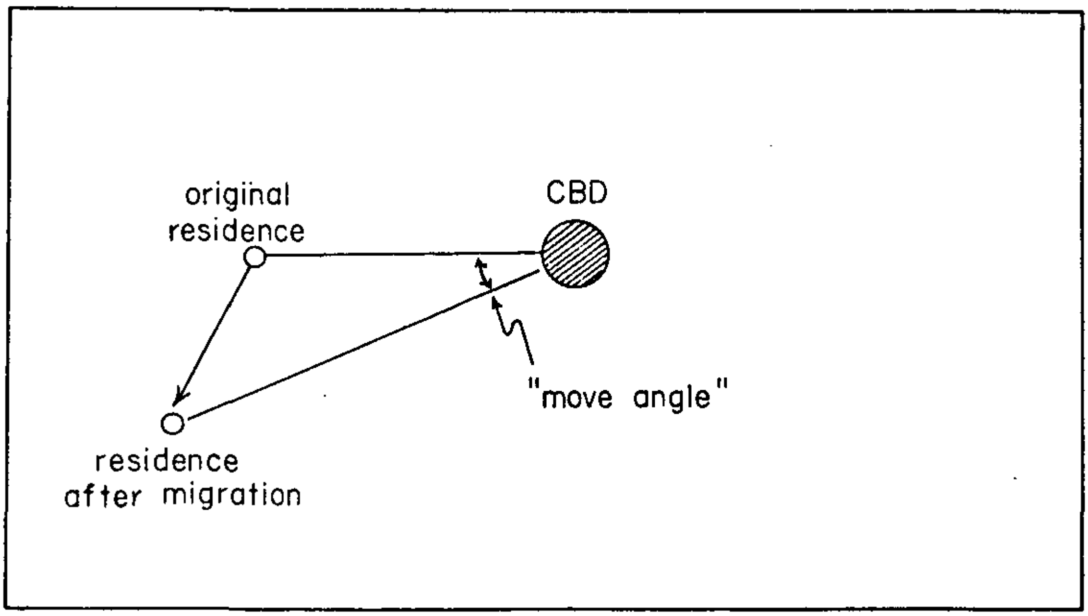

Figure 32 : Definition of the move angle.

Finite Difference Solution. Since the permeabilities in the equation

$$\frac{K_x}{2}\frac{\partial^2 h^2}{\partial x^2} + \frac{K_y}{2}\frac{\partial^2 h^2}{\partial y^2} + (\beta-\delta-\omega)\ h = \frac{\partial h}{\partial t} \quad . \ . \ . \ . \ . \ . \ . \ [\ 8.13\ ]$$

are not constant over an urban region, but vary with the topography, transportation network etc., analytical solution for real-world application is not feasible. The partial derivatives may, however, be approximated by finite differences, and a solution for $h(x,y,t)$ obtained by numerical computation on a digital computer.

The non-linearity of Eq.[8.13] precludes implicit solution techniques, since the resulting set of simultaneous equations are quadratic. An explicit solution algorithm, however, is readily formulated by representing the term $\partial h/\partial t$ by a forward difference, the derivatives in h^2 by central differences, namely

$$\frac{\partial h}{\partial t} = \frac{h^{t+1}_{i,j} - h^{t}_{i,j}}{\Delta t}$$

$$\frac{\partial^2 h^2}{\partial x^2} = \frac{(h^t_{i-1,j})^2 - 2(h^t_{i,j})^2 + (h^t_{i+1,j})^2}{(\Delta x)^2}$$

$$\frac{\partial^2 h^2}{\partial y^2} = \frac{(h^t_{i,j-1})^2 - 2(h^t_{i,j})^2 + (h^t_{i,j+1})^2}{(\Delta y)^2}$$

Where $h^t_{i,j}$ is the population density at the point x = i, y = j, at time t. Using these finite differences expression [8.13] may be written as

$$\frac{h^{t+1}_{i,j} - h^t_{i,j}}{\Delta t} = \frac{K_{x(i,j)}}{2} \quad \frac{(h^t_{i-1,j})^2 - 2(h^t_{i,j})^2 + (h^t_{i+1,j})^2}{(\Delta x)^2}$$

$$+ \frac{K_{y(i,j)}}{2} \quad \frac{(h^t_{i,j-1})^2 - 2(h^t_{i,j})^2 + (h^t_{i,j+1})^2}{(\Delta y)^2}$$

$$+ (\beta-\omega-\delta)\, h^t_{i,j} \quad \ldots\ldots\ldots \quad [\ 8.21\ ]$$

resulting in the explicit solution for $h^{t+1}_{i,j}$ (i.e. in terms involving only h^t)

$$h^{t+1}_{i,j} = \frac{\Delta t\ K_{x(i,j)}}{2} \quad \frac{(h^t_{i-1,j})^2 - 2(h^t_{i,j})^2 + (h^t_{i+1,j})^2}{(\Delta x)^2}$$

$$+ \frac{\Delta t\ K_{y(i,j)}}{2} \quad \frac{(h^t_{i,j-1})^2 - 2(h^t_{i,j})^2 + (h^t_{i,j+1})^2}{(\Delta y)^2}$$

$$+ (1 + \Delta t\ (\beta-\delta-\omega))\, h^t_{i,j} \quad \ldots\ldots\ldots \quad [\ 8.22\ ]$$

Richtmeyer and Morton (101) suggested a heuristic approach to the problem of estimating the stability criterion for such a non-linear equation. By writing the general non-linear differential equation

$$\frac{\partial h}{\partial t} = K\, \frac{\partial^2 h^n}{\partial x^2}$$

in the form

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} (K\ n\ h^{n-1} \frac{\partial h}{\partial x})$$

and comparing this formulation with the corresponding linear equation

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}(K \frac{\partial h}{\partial x}) \quad \text{. . . . . [8.23]}$$

it is evident that the "effective" diffusion coefficient for Eq.[8.23] may be written as $K\ n\ h^{n-1}$. Substituting this for K in the well-known expression for the stability criterion of the linear form, one obtains[9]

$$\frac{K\ n\ h^{n-1}\ \Delta t}{(\Delta x)^2} \leq 0.5$$

In direct analogy to the two-dimensional linear equation

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} = \frac{\partial h}{\partial t}$$

for which the stability criterion for explicit solution is

$$\frac{K_x\ \Delta t}{(\Delta x)^2} + \frac{K_y\ \Delta t}{(\Delta y)^2} \leq 0.5$$

---

[9]Adrian and Lo (102) derived this condition for the one-dimensional non-linear equation for unconfined groundwater flow by comparison of the finite difference expressions.

$$K = K^* \left(\frac{\text{specific weight}}{\text{dynamic viscosity}} \right)$$

The permeability of the urban growth model may thus be visualized as the product of a constant physiographic permeability k* and a mobility factor ψ that may vary in time (and from population to population).

Dimensional analysis of the Darcy Law Analogy yields

$$u \frac{[L]}{[T]} = - K_{x,y} \frac{[L]^4}{[T][M]} \frac{dh}{dx} \frac{[M]}{[L]^2[L]}$$

postulating dimensions of $[L]^2/[T]$ for the mobility factor ψ

$$K_{x,y} \frac{[L]^4}{[T][M]} = K^* \frac{[L]^2}{[M]} \psi \frac{[L]^2}{[T]}$$

hence the intrinsic permeability has dimensions $[L]^2/[M]$, intuitively plausible as a measure of saturation density (as acres per person). The logic of terming the non-constant part of overall permeability as mobility factor is apparent from the dimensions $[L]^2/[T]$, namely acres per year; thus representing the potential rate of residential expansion.

Insofar as the physical groundwater analogy involves the flow of matter radially from the center, outwards to an undisturbed periphery, a specific set of behavioural assumptions is implied by the urban diffusion model. The postulated relationship between density gradient and migration does not appear in conflict with currently one obtains for Eq.[8.22] the stability condition

$$\frac{2\ K_x\ h\ \Delta t}{(\Delta x)^2} + \frac{2\ K_y\ h\ \Delta t}{(\Delta y)^2} \leq 0.5 \qquad [\ 8.24\ ]$$

Richtmeyer and Morton (101) have also shown that stability is unaffected by the lower order terms. By suitable scaling of h, stability is easily attained, since an approximate prior estimate of the maximum population density that will occur during the computation is generally possible.

To evaluate the boundary condition [8.18] we may express the derivative as

$$\frac{\partial h}{\partial r} = \frac{h_{r_w} - h_{r_w + \Delta r}}{\Delta r} \qquad [\ 8.25\ ]$$

hence Eq.[8.18] becomes

$$h_{r_w}\ (h_{r_w} - h_{r_w + \Delta r}) = \frac{\mu\ \Delta r}{2\ \pi\ r_w\ K} \qquad [\ 8.26\ ]$$

which is quadratic in h_{r_w} with solution

$$h_{r_w} = \frac{h_{r_w+\Delta r} \pm \sqrt{h^2_{r_w+\Delta r} + \frac{2\ \mu\ \Delta r}{\pi\ r_w\ K}}}{2} \qquad [\ 8.27\ ]$$

For $\mu > 0$, it is clear that $h_{r_w} > h_{r_w+\Delta r}$ and hence the positive term of Eq.[8.27] gives the required value of h_{r_w}. This expression is readily generalised to two dimensions by using average values for K and r_w. If (i,j) are the coordinates of the in-migration point, we may replace h_{r_w+r} by h_*^t and K by K_* in Eq.[8.27], where

$$h_*^t = \frac{h_{i+1,j}^t + h_{i-1,j}^t + h_{i,j+1}^t + h_{i,j-1}^t}{4}$$

and

$$K_* = \frac{K_{y(i+1,j)} + 2\,K_{y(i,j)} + K_{y(i-1,j)}}{8}$$

$$+ \frac{K_{x(i,j+1)} + 2\,K_{x(i,j)} + K_{x(i,j-1)}}{8}$$

Figures 33 and 34 show computed density profiles for a hypothetical urban area of isotropic, homogenous intrinsic permeability. The effect of an increase in the mobility factor ψ (lower central density, decrease in density gradient) describes adequately the phenomenon of urban sprawl. Keeping the intrinsic permeability and mobility factor constant, population growth alone results in a diminished density gradient, corresponding to the adaptation to growth by decrease in concentration described by Winsborough (92).

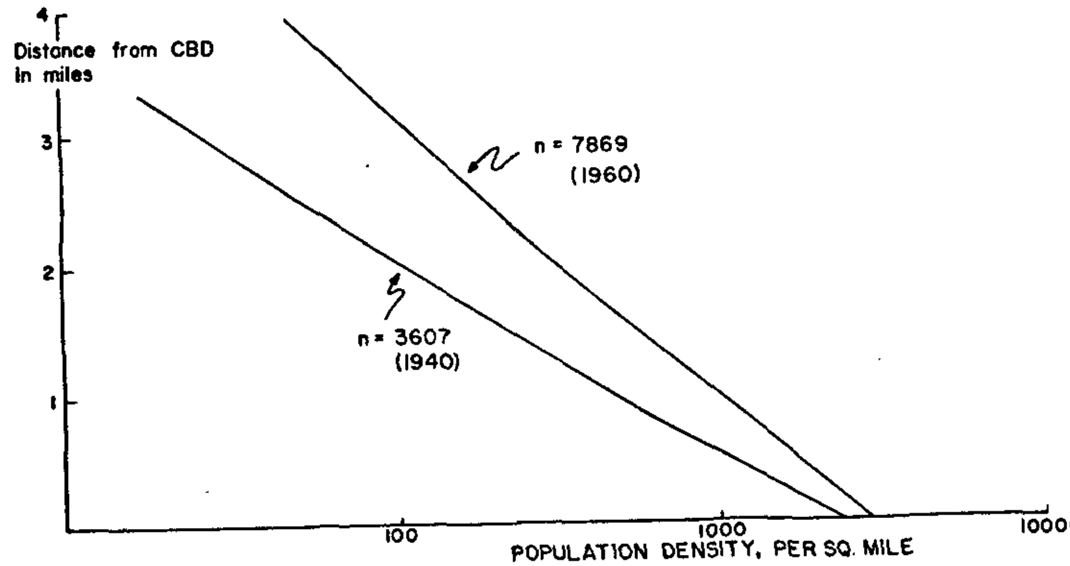

Figure 33 : Relation of the density gradient to population growth under constant permeability for a hypothetical city

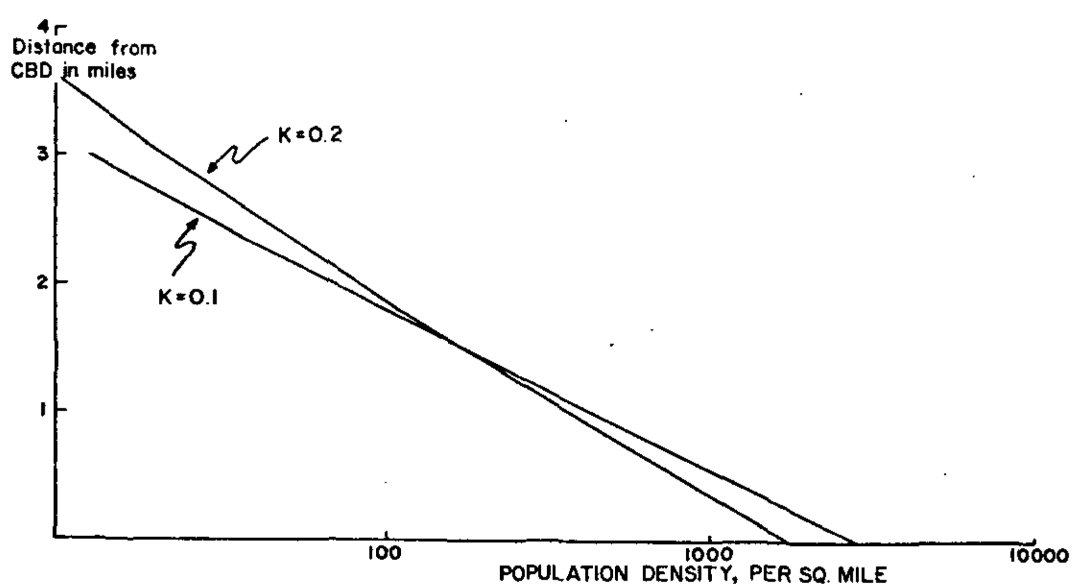

Figure 34 : Density gradients under varying permeability constants for a hypothetical city

CHAPTER IX

PROJECTION OF SERVICE AREAS

Current practice. A review of engineering reports shows that specification of the service area population is generally even more arbitrary than the population projection itself/ In cases of existing systems, for which treatment capacity expansion is proposed, the most common statement is " we have estimated that the sewered population will be X by 19XX",(or some words to this effect), generally a greater fraction than presently serviced but less than universal. Where a treatment facility is planned in conjunction with a new collection system, the analysis is generally more thorough, with serviced populations based on probable fractions of saturation densities under existing zoning regulations. The saturation density method of population projection, popular in Master Plans of the Fifties, appaers to have fallen from favour, yielding improbably high results. A more rational basis for service area projection is desirable.

Application of the diffusion model. The diffusion model of residential development outlined in the previous chapter has obvious potential as a projection device in conjunction with the stochastic simulation population projection of Chapter VII, since the later yields the birth, death and migration rates required as the driving force for the diffusion model. An initial condition can be obtained from a land-use study or census base maps (or even aerial photographic surveys) that indicate existing locations of residential land use.

The basis for inclusion of a particular cell into the water or sewer service area is the relationship between prevailing residential density and distance from the existing area. Analytical formulation of a decision rule to include or exclude a particular cell is generally quite complex. Downing (103) has explored this approach using some available treatment plant cost functions and evaluating the relationships between incremental costs at the treatment plant, incremental transportation costs to the plant, and the estimated benefits of sewer access to the individual householder. Although the distance-density trade-offs between sewers and septic tanks could be identified in a general way, local conditions induce considerable variability to such estimates. An individual economic analysis for each and every cell at all time points during numerical computations would not only require inordinate amounts of computer time but demand that the marginal costs be known. At present the availability of sufficiently accurate cost information for a particular treatment plant is improbable.

Empirical expedients are, fortunately, sufficient. The extent of the present service areas is generally known, and an examination of land use, zoning and topographic maps will yield an estimate of the marginal residential density at the existing peripheral limits of the service area. The tedium of this operation is eliminated by computerization. By examination of the distances to the service area of cells <u>not</u> presently serviced, an appropriate density-distance decision rule may be specified. At each time point during the numerical solution of the diffusion equations, cells in the vicinity of the current service area are evaluated on the foregoing basis.

The computer model. Although the numerical solution of Eq. [8.13] by the finite difference algorithm [8.21] can be compactly programmed, the numerous subsidiary computations required to translate the mathematical abstraction into a scale model of the real world make the complete program somewhat complicated. The links between individual subroutines are indicated on Figure 35, and the computer model is best described on a routine by routine basis using this Figure as an outline.

Subroutine INPUT1 converts a map of physiographic characteristics into a basic matrix of intrinsic permeability. State forest, wetlands and conservation areas, major rivers, lakes and topographical obstacles are assigned zero permeability, since no residential growth will occur in such cells. The maximum grid size is presently 40 x 40. Cells outside the immediate problem boundary (few communities are rectangular) are also assigned zero permeability. Cells adjacent to lakes are automatically assigned high permeabilities, as are cells along the transportation network. Permeabilities along the latter are directional: a north-south road, for example, receives a higher Y-permeability than X-permeability. The input-physiographic map also contains information on the existing residential land use, as obtained

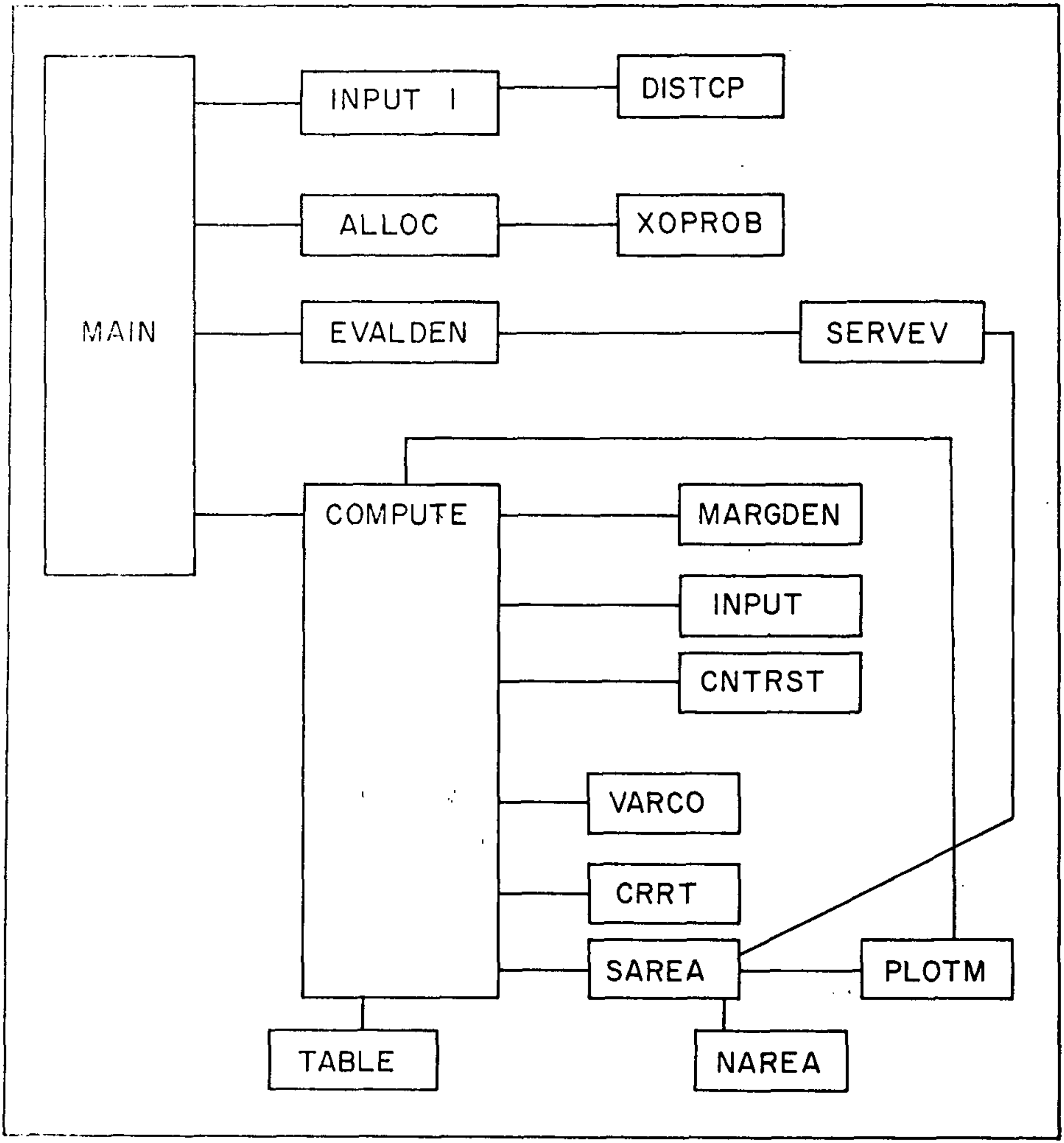

Figure 35 : Flow chart, residential growth simulation model (PROGRAM SIMGR)

from the Land Use Study undertaken for the Lower Pioneer Valley Regional Planning Commission. This information is in the form of a trichotomous: High-Medium-Low intensity classification. The initial condition of population density is derived from this information. A second input map serves to specify the existing water and sewer service areas. Since there may be more than one center (recharge points) in a given problem, each sewer and water serviced cell is assigned to the center of closest proximity. This computation is executed in subroutine DISTCP. In the sample problem, three centers are specified (Amherst, Granby, Belchertown).

Subroutine ALLOC allocates population to cells such that consistency with existing service area populations is attained. Routine XDPROB specifies the rules (e.g. negative exponential decline from the center) by which additional service area population is allocated. Hence the need to associate each serviced cell to its nearest centre. Population is allocated outside the service area by drawings from uniform random number tables: a cell requires some number of hits that is inversely related to its intrinsic permeability before a household is allocated to it. In this fashion, a rural household is more likely to be allocated to a roadside or lakeside location than to an inaccessible cell. This allocation process is illustrated by Table 22.

	Actual	Allocated by INPUT1 as per land use map	Allocated by ALLOC	
			neg.exp.	uniform random
Water Service Area	13100	7520	5580	0
Sewer Service Area	10300	4320	5980	0
Outside Service Areas	10770	5888	0	4882

Table 22 : Allocation of residential population in the initial condition.

Subroutine EVALDEN computes the mean density (and lower moments) of water and sewer serviced cells, and SERVEV evaluates the serviced population. This latter function is not trivial, since population density is recorded in one array and the instantaneous spatial location of the water and sewer service areas identified by an integer in a second array (0 = Unserviced, 1 = Water serviced, 2 = Sewer serviced).

Subroutine MARGDEN then computes the existing marginal density at the periphery of the service areas, together with the associated lower moments. These marginal densities serve as the basis for the decision-rule for inclusion or exclusion of cells adjacent to the currently existing service area as the iteration proceeds. The average equilibrium radius is also evaluated at this point, being equated in rough approximation to the periphery of the water service area.

INPUT2 processes the output from the population projection program POPPRJ (see Chapter VII). Subroutine CNTRST allows further manipulation of the basic intrinsic permeability matrices. These are specified initially on an integer scale 0-9.

COMPUTE contains the actual core algorithm, the numerical solution of the finite difference expressions. The problem is automatically scaled to ensure stability. The call to CRRT corrects for truncation error at each iteration. This correction is possible since the total in the system is known from the population projection. The correction at each iteration is in the order of 0.01-1%, dependent on the distance and time scales involved. Though small, they are cumulative, resulting in noticeable losses for uncorrected computations.

A further source of loss is the presence of zero permeability cells (and hence zero population density) in the body of the problem region. In actuality, at such points the density gradient is undefined (e.g. at a lakeside), and the use of "image" points is a convenient device to attain zero gradient. Suppose the location (i, j + 1) has zero permeability. Then for evaluation of $h_{i\ j}$, the term $h_{i,\ j+1}$ in the expression [8.21] is zero. This is replaced by $h_{i,j}$ in evaluating [8.21]. The call to VARCO evaluates the variable coefficients at each time-point (instantaneous birth, death and migration rates).

PLOTM is the plotting routine, called at 5-year intervals. A summary table (Routine TABLE) is printed out on completion of the computations.

The output of a sample run is reproduced on the following pages, showing the projected expansion of service areas for the north-

SERVICE AREA MAP 1965

```
***************************************
*         X                           *
*         XXX                         *
*       YYY                           *
*     XYYYY                           *
*       Y Y                           *
*       Y Y                           *
*       Y Y                           *
*       Y Y                           *
*       YYY  YYYXX                    *
*        YYYYY                        *
*        YYYYY                        *
*        YY  YY                       *
*       YY    X                       *
*        Y    XX                      *
*        Y                            *
*        YY                           *
*        YYYYYXX                      *
*        X   X                        *
*        X   X X                      *
*        X   X X                      *
*        X   XXXX                     *
*                                     *
*                                     *
*                      XX             *
*                        X            *
*              XX         X           *
*           XXX  X         Y          *
*                           YYYY      *
*                          YYY        *
*    XX        X           YYY        *
*    X XX    X X             YY       *
*    X   X XXXXX               Y      *
*    XX    XXX                        *
*    XXXX  X X                        *
*       XXX  X                        *
*       X  X  X                       *
*       XX X                          *
*       X XX                          *
***************************************
*                                     *
* X EXISTING WATER SERVICE AREA       *
* Y EXISTING SEWER SERVICE AREA       *
* W WATER SERVICE AREA ADDITIONS      *
* S SEWER SERVICE AREA ADDITIONS      *
* TOTAL POPULATION           23870    *
* SEWERED POPULATION         10300    *
* POP.IN WATER SERVICE AREA  13150    *
* FRAC.SEWERED                0.43    *
* FRAC.IN WATER SERVICE AREA  0.55    *
*                                     *
***************************************
```

RESIDENTIAL LAND USE INTENSITY MAP 1965

```
****************************************
*                                      *
*          +                           *
*      $+$$+                           *
*      -$$$$                           *
*        $ $                           *
*        $ $             -++           *
*        $ $+              +           *
*     ++$++$                           *
*        $$$$+$$$---                   *
*        +$$$$$     -++                *
*        $$$$$$                        *
*         $$  $$                       *
*        +$                            *
*         $                            *
*         $               +            *
*         $$    -        ++            *
*         +$$+$$                       *
*              -                       *
*         +                            *
*              +      - -   +          *
*         -+          +                *
*                     ++               *
*                    +                 *
*                                      *
*                        +-            *
*              +          +            *
*           -    -         $           *
*      +                   $+$$        *
*     +                   +$$   -      *
*     -        -          $$$      -   *
*    +  - -  -   +    +--   $$         *
*        - -++- -             -        *
*    --  - -++                         *
*    +-+                               *
*     $ ---  -                         *
*       -  -   -                -      *
*       -                        +-    *
*       - ---                     -    *
****************************************
*                                      *
* $ HIGH DENSITY .GT. 4 DWELLINGS/ACRE *
* + MEDIUM DENSITY  2-4 DWELLINGS/ACRE *
* - LOW DENSITY  .LT. 2 DWELLINGS/ACRE *
*                                      *
* SCALE - 1 CELL =16 ACRES             *
*                                      *
****************************************
```

Figure 36A : Service area projection, Northeastern periphery of the LPVRPD

SERVICE AREA MAP 1980

```
***************************************
*          XW                         *
*      WWW S                          *
*      SSYYYSW                        *
*      SYYYYSW                        *
*      SSYSYSW                        *
*        YSYSS                        *
*        YSYSSW                       *
*        YSYSSSSSW                    *
*        YYYSSYYYSXWW                 *
*        SYYYYYSSWXWWW                *
*        SYYYYYS  X                   *
*        SYY  YYW                     *
*        YY   SXW                     *
*         YS   XX                     *
*        YYS                          *
*        SYYWWWW                      *
*        WYYYYYYXX                    *
*        WXWWWXWW                     *
*         XW  XWX                     *
*         XW  XWX                     *
*         X   XXXX      W             *
*                      WWW            *
*                      W W            *
*                        XX           *
*                          XW         *
*               XX         WXW        *
*            XXX  X         SYSSS     *
*      W                    SYYYYW    *
*     W                    WYYYSSW    *
*    SXW        X          WYYYS      *
*    SSXXWW WXWX           WSSYYW     *
*    SSSWXWXXXXX            W SSY     *
*    SSSSWWXXXW                       *
*    SSSSSWXWX                        *
*     SSSSXW X                        *
*     SSSWWX  X                       *
*       XXWX                          *
*       XWXX                          *
***************************************
*                                     *
* X EXISTING WATER SERVICE AREA       *
* Y EXISTING SEWER SERVICE AREA       *
* W WATER SERVICE AREA ADDITIONS      *
* S SEWER SERVICE AREA ADDITIONS      *
* TOTAL POPULATION            35000   *
* SEWERED POPULATION          26390   *
* POP.IN WATER SERVICE AREA   32130   *
* FRAC.SEWERED                 0.75   *
* FRAC.IN WATER SERVICE AREA   0.92   *
*                                     *
***************************************
```

RESIDENTIAL LAND USE INTENSITY MAP 1980

```
****************************************
*         --                           *
*     +++ ++-                          *
*     ++++++-                          *
*     +$$$$+-                          *
*     $$$$$$+                          *
*       $$$$$              ---         *
*       $$$$$+                         *
*       $$$$$$++-                      *
*       $$$$$$$++-                     *
*       $$$$$$$++-                     *
*       $$$$$$$                        *
*       $$$  ++-                       *
*       $$   ++-                       *
*        $$   -                        *
*       +$+                            *
*       ++++---                        *
*       +++++----                      *
*        ++-----                       *
*        --  ---                       *
*        --  ---                       *
*                        -             *
*                       ---            *
*                                      *
*                                      *
*                             --       *
*              --             -++      *
*                              +$$++   *
*                              +$$$+-  *
*     -                       +$$$$+-  *
*    ++                       +$$$$    *
*    $++-   ----              -+$$+-   *
*    $$$+-------               - ++-   *
*    $$$$+-----                        *
*    $$$$+----                         *
*     $$$+-- -                         *
*     $$++-                            *
*       --                             *
*       ---                            *
****************************************
*                                      *
* $ HIGH DENSITY   .GT. 4 DWELLINGS/ACRE*
* + MEDIUM DENSITY 2-4    DWELLINGS/ACRE*
* - LOW DENSITY    .LT. 2 DWELLINGS/ACRE*
*                                      *
* SCALE - 1 CELL =16 ACRES             *
*                                      *
****************************************
```

Estimates by the Engineering Consultant to the LPVRPC : Frac.in water service area=0.93, Frac. sewered=0.71

Figure 36B : Service area projection, northeastern periphery of the LPVRPD

```
SERVICE AREA MAP  1990
****************************************
*         WXWW                         *
*      SSS SSXW                        *
*      SSYYYSS                         *
*      SYYYYSSW                        *
*      SSYSYSSS                        *
*        YSYSSSW                       *
*        YSYSSSS                       *
*        YSYSSSSSWW                    *
*        YYYSSYYYSXWW                  *
*        SYYYYYSSSXWWW                 *
*        SYYYYYSSWX                    *
*        SYY  Y S                      *
*        YY   SSW                      *
*         YSW WXX                      *
*        SYSW  W                       *
*        SYYSWWWW                      *
*        SYYYYYYXX                     *
*        SSSWWXWW                      *
*        WXWWWXWX                      *
*         XW  XWX                      *
*         XWWWXXXX    WW               *
*                     WWWW             *
*                   WWW W              *
*                       XXW            *
*                         XW           *
*               XXW       WSS          *
*            XXXWWX        SYSSSW      *
*        W                 SYYYYSW     *
*       SW                SYYYSSSW     *
*      SSSW      X        SYYYSS       *
*      SSSSWWWWXWX       WWSSYYS       *
*      SSSSSWXXXXX        WWWSSYW      *
*      SSSSSSXXXW              W       *
*      SSSSSSXWXW                      *
*       SSSSSWWXW                      *
*       SSSSSXWWX                      *
*       WSSSWX                         *
*        WSWXX                         *
****************************************
*                                      *
* X EXISTING WATER SERVICE AREA        *
* Y EXISTING SEWER SERVICE AREA        *
* W WATER SERVICE AREA ADDITIONS       *
* S SEWER SERVICE AREA ADDITIONS       *
* TOTAL POPULATION             47700   *
* SEWERED POPULATION           40470   *
* POP.IN WATER SERVICE AREA    45730   *
* FRAC.SEWERED                  0.85   *
* FRAC.IN WATER SERVICE AREA    0.96   *
*                                      *
****************************************
```

```
RESIDENTIAL LAND USE INTENSITY MAP  1990
****************************************
*        -+--                          *
*     +++ +++-                         *
*     $$$$$$+-                         *
*     $$$$$$$-                         *
*     $$$$$$$+                         *
*       $$$$$$-         --             *
*       $$$$$$$                        *
*       $$$$$$$++-                     *
*       $$ $$$$$+-                     *
*       $$$$$$$$+-                     *
*       $$ $$$$$--                     *
*       $$$  $$+                       *
*       $$   ++-                       *
*        $$-  --                       *
*       $$$-  -                        *
*       $$$++--                        *
*       $$+++---                       *
*       ++++----                       *
*        --- ---                       *
*        --  ---                       *
*        -   ---     -                 *
*                    --                *
*                                      *
*                                      *
*                       --             *
*              --       -++-           *
*                        +$$$+-        *
*                        $$$$$+        *
*     +-                ++$$$$+        *
*    $$+-               +$$$$$ -       *
*    $$$++------       -++$$$+         *
*    $$$$++----          +-+++         *
*    $$$$$+----                        *
*    $$$$$+----                        *
*     $$$$+----                        *
*     $$$++-                           *
*     -+++--                           *
*      -++--                           *
****************************************
*                                      *
* $ HIGH DENSITY .GT. 4 DWELLINGS /ACRE *
* + MEDIUM DENSITY 2- 4 DWELLINGS/ACRE *
* - LOW DENSITY  .LT. 2 DWELLINGS/ACRE *
*                                      *
* SCALE - 1 CELL =   ACRES             *
*                                      *
****************************************
```

Estimates by the Engineering Consultant to the LPVRPC : Frac.in water service area = 0.95, Frac. sewered = 0.75

Figure 36C : Service area projection, northeastern periphery of the LPVRPD eastern part of the Lower Pioneer Valley Regional Planning District. The birth, death and migration rates were obtained from the projection Model B of Chapter VII.

So far we have not considered actual numerical values of intrinsic permeability or the mobility factor. The 1990 projections of Figure 37 for East Longmeadow will illustrate the sensitivity of the model to particular numerical values.

The basic matrix of intrinsic permeability is specified on an integer scale 0-9, and multiplied by a factor π to obtain a matrix of greater contrast. Thus the projection of Figure 37A, for example, used $\pi = 3.0$ and a mobility factor of 0.2 to give an effective maximum permeability of

$$9 \times 3.0 \times 0.2 = 13.5 = K_{max}$$

To ensure stability, this particular problem was automatically scaled down by 10^{-4} (equivalent to specifying population in tens of thousands).

The effect of allowing changes in the peripheral density that determine whether or not a cell will be accessed by the service area is also illustrated. The present marginal density for sewer and water access is denoted by h(s), h(w) respectively, and the marginal density in the terminal year given by $\theta_s h(s)$, $\theta_w h(w)$ where $\theta \leq 1$. Figure 37A assumes θ_w, $\theta_s = 1$ (no change) whereas Figure 37B takes θ_w, $\theta_s = 0.65$, on the assumption that sewers and water lines

E.LONGMEADOW


```
SERVICE AREA MAP  1990
****************************************
*                                      *
*                                      *
*           WXXXW XXX                  *
*   SSS  YS SXXWW   X X                *
*  SYYY  YSSSSWW    XX X               *
*  SYYYYYYSSSSSW    X   XX             *
*  SSSSSYYYSYSSSW XXX     X            *
*  WSSSYYYYSYSYYSX  X                  *
*   SSSYYSYYYYSYYW   X                 *
*   SSSYYYYYY STSY   X                 *
*      YYYYYYYYSSY    X                *
*     SYYYYYYYYSSWY    X               *
*     SYYYYYYYSSSWY     X              *
*   SSSYYYYYYY SWWY      X             *
*   SSSSSSYSYSYYSWX                    *
*   SSSSSYSSSSSSSXXW   WXXX            *
*    SSSYYYYYYSSWXXWWWXXXWX            *
*    WSSYYSSSSSW  XXXXXXXWX            *
*     WYYYSSSXW   XWWWXXXWX            *
*     WYYYSW  X    WWWXXXXX            *
*    WWYYYWW   X  XXX                  *
*    SSYYYYW   XXXX                    *
*     SS WYYYYX                        *
*     S WWW                            *
*     S WW                             *
*     SSSW                             *
*                                      *
*                                      *
****************************************
*                                      *
* MOBILITY FACTOR = 0.2                *
* PI = 3.0                             *
* THETA(S) = 1.0                       *
* THETA(W) = 1.0                       *
* XSCALE = 10E-04                      *
*                                      *
****************************************
*                                      *
* X EXISTING WATER SERVICE AREA        *
* Y EXISTING SEWER SERVICE AREA        *
* W WATER SERVICE AREA ADDITIONS       *
* S SEWER SERVICE AREA ADDITIONS       *
* TOTAL POPULATION            19300    *
* SEWERED POPULATION          13204    *
* POP.IN WATER SERVICE AREA   16563    *
* FRAC.SEWERED                 0.68    *
* FRAC.IN WATER SERVICE AREA   0.86    *
*                                      *
****************************************
```

Figure 37A : Service area projection
East Longmeadow

```
E.LONGMEADOW
SERVICE AREA MAP 1990
*****************************************
*                                       *
*                                       *
*            SXXXWWXXX                  *
*   SSS  YS SSXWW   X X                 *
*  SYYY  YSSSSSW    XX X                *
*  SYYYYYYSSSSSSW   X   XX              *
*  SSSSSYYYSYSSSWWXXX     X             *
*   SSSYYYYSYSYYSX  X                   *
*   SSSYYSYYYYSYYW   X                  *
*   WSSYYYYYY SYSY   X                  *
*      YYYYYYYYYSSY   X                 *
*     SYYYYYYYYSSWY    X                *
*     SYYYYYYYYSSW Y    X               *
*   SSSYYYYYYY SWWY      X              *
*   SSSSSSYSYSYYWWX     WW              *
*   SSSSSYSSSSSSSSXXW  WXXX             *
*    SSSYYYYYYYSSWXXWWWXXXWX            *
*     SSYYSSSSSW  XXXXXXXWX             *
*     WYYYSSSXW   XWWWXXXWX             *
*     WYYYS   X    WWWXXXXX             *
*    WSYYYW    X  XXX                   *
*    SSYYYYW   XXXX                     *
*     SS WYYYYX                         *
*     S WWW                             *
*     S WW                              *
*     SSSW                              *
*                                       *
*                                       *
*****************************************
*                                       *
* MOBILITY FACTOR = 0.2                 *
* PI = 4.0                              *
* THETA(S) = 0.66                       *
* THETA(W) = 0.66                       *
* XSCALE = 10E-04                       *
*                                       *
*****************************************
*                                       *
* X EXISTING WATER SERVICE AREA         *
* Y EXISTING SEWER SERVICE AREA         *
* W WATER SERVICE AREA ADDITIONS        *
* S SEWER SERVCIE AREA ADDITIONS        *
* TOTAL POPULATION            19300     *
* SEWERED POPULATION          13428     *
* POP.IN WATER SERVICE AREA   16744     *
* FRAC SEWERED                 0.70     *
* FRAC.IN WATER SERVICE AREA   0.87     *
*                                       *
*****************************************
```

Figure 37B : Service area projection
East Longmeadow

```
E. LONGMEADOW
SERVICE AREA MAP  1990
****************************************
*                                      *
*                                      *
*            WXXXW XXX                 *
*    SSS  YS WXXWW   XWX               *
*   SYYY  YS  XWW    XX X              *
*   SYYYYYYSSSSWW    X   XX            *
*   WSSSSYYYSYSSWW XXX     X           *
*    SSSYYYYSYSYYWY  X                 *
*    SSSYYSYYYYSYYW   X                *
*    WSSYYYYYY SYSY   X                *
*       YYYYYYYYYSSY    X              *
*      SYYYYYYYYYSWWY    X             *
*      SYYYYYYYYSS  Y     X            *
*    WSYYYYYYYYY W  Y      X           *
*    SSSSSSYSYSYYWWX     WW            *
*    SSSSSYSSSSSSXXX   WWXXX           *
*     WSSYYYYYYYSXWXXWWWXXXWX          *
*      WSYYSSSSW   XXXXXXXWX           *
*      WYYYSWWX    XWWWXXXWX           *
*      WYYYS   X     WWXXXXX           *
*      WYYYW    X  XXX                 *
*     WSYYYYW   XXXX                   *
*      SS  YYYYX                       *
*      S                               *
*      S S                             *
*      SSS                             *
*                                      *
*                                      *
****************************************
*                                      *
* MOBILITY FACTOR = 0.1                *
* PI = 3.0                             *
* THETA(S) = 0.66                      *
* THETA(W) = 0.66                      *
* XSCALE = 1oE-04                      *
*                                      *
****************************************
*                                      *
* EXISTING WATER SERVICE AREA X        *
* Y EXISTING SEWER SERVICE AREA        *
* W WATER SERVICE AREA ADDITIONS       *
* S SEWER SERVICE AREA ADDITIONS       *
* TOTAL POPULATION            19300    *
* SEWERED POPULATION          12933    *
* POPULATION IN WATER SERVICE 16642    *
* FRAC SEWERED                 0.67    *
* FRAC. IN WATER SERVICE AREA  0.86    *
*                                      *
****************************************
```

Figure 37C : Service Area projection
East Longmeadow

E.LONGMEADOW

SERVICE AREA MAP 1990

```
*******************************************
*                                         *
*                                         *
*              XXX  XXX                   *
*         Y   XX     X X                  *
*   YYY   Y   X      XX X                 *
*   YYYYYY    X      X   XX               *
*   XSSSYYYSY      XXX     X              *
*    SSYYYYSYSYY X   X                    *
*    SSYYSYYYY YY     X                   *
*    SSYYYYYYY SY Y   X                   *
*      YYYYYYYY   Y    X                  *
*     SYYYYYYYYS   Y    X                 *
*     SYYYYYYYS    Y     X                *
*    WSYYYYYYY     Y      X               *
*   XXSSSSYSYSYY   X                      *
*    XSSSYSSSSW XXX      XYX              *
*     XSYYYYYY X XX   WXXX X              *
*      WYYSSX      XXXXXXX X              *
*      YYYW   X    X  WXXX X              *
*      YYY     X       XXXXX              *
*      YYY      X  XXX                    *
*      YYYY     XXXX                      *
*     SS  YYYYX                           *
*     S                                   *
*     S                                   *
*     SS                                  *
*                                         *
*                                         *
*                                         *
*******************************************
*                                         *
* MOBILITY FACTOR = 0.1                   *
* PI = 3.0                                *
* THETA(S) = 1.0                          *
* THETA(W) = 1.0                          *
* XSCALE = 10E-04                         *
*                                         *
*******************************************
*                                         *
* X EXISTING WATER SERVICE AREA           *
* Y EXISTING SEWER SERVICE AREA           *
* W WATER SERVICE AREA ADDITIONS          *
* S SEWER SERVICE AREA ADDITIONS          *
* TOTAL POPULATION              19300     *
* SEWERED POPULATION            12596     *
* POP.IN WATER SERVICE AREA     15026     *
* FRAC.SEWERED                   0.65     *
* FRAC.IN WATER SERVICE AREA     0.78     *
*                                         *
*******************************************
```

Figure 37D : Service area projection
East Longmeadow

will be extended to areas of lower density than at present.

While the estimates are quite comparable to those of the planning consultant, the use of the simulation model does allow immediate quantitative analysis of particular assumptions, which cannot be considered explicitly in a traditional estimate.

Further developments. In the interregional projection model of Chapter VII the average lot size was required during each time period of the projection. By including an array of minimum lot sizes, as rquired by zoning regulaticns, the average lot size may be obtained from the residential location simulation model by comparison of cell occupancy with the corresponding matrix of lot sizes. Since the fraction sewered at each time point is also required for the inter-regional projection, a desirable development would be the integration of the two programs to run simultaneously. This in turn demands a much larger grid than the present 40 x 40 maximum, and thus the programs will require major revision to allow intermediate storage of matrices on tape. However, a simple iterative scheme was found adequate for the East Longmeadow example; the projection program POPC assumed a set of values for FRACSEW, the output population projections then inserted into the residential development simulation program SIMGR; a second run of POPC using the second estimate of FRACSEW from SIMGR; satisfactory convergence was obtained in two or three runs.

Evaluation of the model. Despite the many assumptions and approximations that are required to establish a working model, the results appear realistic. The fact that the predicted extent of service areas may be at variance with forecasts obtained by traditional methods does not invalidate the results, since neither can be judged against an absolute standard.

The model does clearly not eliminate any detailed engineering study of proposed service area extensions. However, the results would appear excellent for the purpose of forecasting total flows for the purpose of planning treatment plant facilities and an evaluation of the trade-offs between the scale economies of large regional plants and the costs of regional interceptors. A deterrent to enlarging the scale for detailed projection is the difficulty in specifying permeabilities; the model is thus of doubtful utility for detailed estimates of collection system network requirements. A further limitation of the model is its restricted applicability to the urban periphery. In sparsely settled rural areas assumed centres of inmigration are ill-defined, and the concept of density gradients conceptually vague. Nevertheless, since most treatment facilities of the coming decade are to be built in dynamic suburban areas, this qualification is not serious in an assessment of application potential.

A significant advantage of the mathematical model over traditional expedients is the ability to study the effect of particular assumptions. For example, the effect of decreasing the requisite marginal density for service inclusion may be evaluated by changing a single input parameter. But above all else, the forecast is made on a rational-quantitative basis rather than by intuition and guesswork.

CHAPTER X

CONCLUSIONS AND RECOMMENDATIONS

Conclusions. The principal conclusions to the aforegoing study may be summarized as follows:

1. The response surface optimization technique is an excellent tool for the resolution of computational problems encountered in the decomposition of net migration into in- and out-migration streams. The estimation bias resulting from the stochastic nature of migration and vital rates and the errors in intercensal population estimates may be eliminated for two-region systems by a smoothing scheme for which optimum results are defined by a response surface minimum.

2. The first-order autoregressive stochastic process is a satisfactory model for the quantification of random fluctuations in birth and death rates of local areas. The relationships between the degree of serial correlation and the magnitude of the random fluctuations and the size of the population were found to be statistically significant.

3. Stochastic simulation population projections are demonstrably superior for short-term forecasts to traditional deterministic methods of population projection. Stochastic projections were found to be significantly better for most of the towns in the study region for which preliminary 1970 Census results were available at the time of writing, and in no case was the deterministic forecast significantly better.

4. Stochastic simulation projections permit an objective

and statistically well-defined measure of projection variability that may be directly incorporated in modern treatment plant design procedures. The traditional expedients of specifying "high" and "low" projections are not indicative of actual variability.

5. The model of residential location based on the analogy to a mathematical model of the recharge well was found to be a good representation of the spatial variations in urban population densities, and may be used in an objective assessment of the future extent of water and sewer service areas.

6. The computerized models developed in the course of this study provide a rational basis for the specification of the input to the modern optimization algorithms utilized in the planning of regional waste treatment facilities.

Recommendations. On the basis of this study the following recommendations are directed to the participants in the regional planning process:

1. It is recommended that the economics of time-sharing computer facilities be examined with a view to implementation of computerized planning models.

2. It is recommended that stochastic simulation models be examined as an alternative to traditional methods of population projection.

3. It is recommended that the environmental engineering profession re-evaluate the subordinate roles to which population projection and service area prediction have traditionally been assigned. In particular, population projections need be viewed in the framework of an interregional stochastic process rather than as a qualitative judgement made in isolation of demographic and socio-economic reality.

In addition to the above recommendations, this study suggests the following research priorities:

1. To complete the objective input specification to the optimization algorithms utilized in the planning of regional waste treatment facilities the relationships between serviced population and the stream vector quantity and quality parameters require detailed study. In particular, the traditional loading parameters (for example pounds of BOD and SS per capita per day), institutionalized in State Standards for sewage treatment plant design, require reevaluation in the light of modern conditions.

2. Establish additional computational experience with stochastic simulation population projections and evaluate their performance on a systematic basis on availability of the 1970 Census results.

3. Examine the feasibility of extending the response surface algorithm for the decomposition of migration streams to multi-region systems using additional symptomatic indicators of intercensal population.

4. Extend the service area prediction model to include explicit consideration of zoning regulations. This will probably require mathematical formulation of partially confined aquifer systems as a basis for further development of the physical analogy.

APPENDIX A : STATISTICAL TESTS

Hotelling's T^2 Statistic. To test the hypothesis $\mu = \mu_o$ on the basis of a p-dimensional sample of size N, from a universe distributed as $N(\mu,\Sigma)$, the T^2 statistic, defined as

$$T^2 = N(x - \mu_o)^T S^{-1} (x - \mu_o) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad [\ a.1\ ]$$

may be utilized, where S is the sample covariance matrix, and x the vector of sample means. The critical region is defined for values of T^2 for which

$$T^2 \geq \frac{(N-1)\ p}{N - p} F_{p,N-p,\alpha} = T^2_{cr} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad [\ a.2\ ]$$

where $F_{p,N-p,\alpha}$ is the F-distribution with p, N-p degrees of freedom at significance level α.

To test for the equality of 2 p-dimensional vectors of sample means, i.e. to test the hypothesis $\mu_1 = \mu_2$, based on N_1 and N_2 observations, respectively, a pooled estimate of the sample covariance matrix is first defined as

$$S = \frac{1}{N_1 + N_2 - 2} \{(N_1 - 1)\ S_1 + (N_2 - 1)\ S_2\} \quad . \ . \ . \quad [\ a.3\ ]$$

where S_1 and S_2 are the sample covariance matrices. Then if y_1 and y_2 are the sample means

$$T^2 = \frac{N_1 N_2}{N_1 + N_2}(y_1 - y_2)^T\ S^{-1}\ (y_1 - y_2) \quad . \ . \ . \ . \ . \ . \quad [\ a.4\ ]$$

which is distributed as T^2 with $N_1 + N_2 - 2$ degrees of freedom. The critical region is defined for values of T^2 greater than

$$T^2_{cr} = \frac{(N_1 + N_2 - 2)}{(N_1 + N_2 - p - 1)} F_{p,N_1+N_2-p-1,\alpha} \quad . \quad . \quad . \quad . \quad . \qquad [\ a.5\ ]$$

Multivariate Equality of Variance-Covariance Matrices. To test for the equality of a set of variance-covariance matrices, let S^k be the estimated covariance matrix of the k-th sample, k=1,2,...,K, each based on n_k observational p-tuples. Defining the pooled estimate of variance, S, based on all $N_1 + N_2 + \ldots + N_K = N$ observations, such that $s_{ij} \in S$ are given by

$$s_{ij} = \frac{1}{N-K} \sum_{k=1}^{K} (N_k - 1)\ s^k_{ij} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad [\ a.6\ ]$$

Anderson (105) has shown that the expression

$$-2\ \rho\ \log_e \frac{\prod^{K}(|S^k| \exp(N_k/2))}{(|S| \exp\{(N-K)/2\})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad [\ a.7\ ]$$

is distributed as chi-square with p(p+1)(K-1) degrees of freedom, and where ρ is defined as

$$\rho = [1 - \{\sum^{K} \frac{1}{N_k - 1} - \frac{1}{N-K}\} \frac{2p^2+3p-1}{6(p+1)(K-1)}] \quad . \quad . \quad . \quad . \quad . \quad . \qquad [\ a.8\ ]$$

APPENDIX B : ESTIMATION BY MINIMUM ABSOLUTE DEVIATIONS (MAD)

Given the regression problem

$$\underset{(n \times 1)}{y} = \underset{(n \times k)}{W} \; \underset{(k \times 1)}{\gamma} \quad \underset{(n \times 1)}{u^t} \quad \ldots\ldots\ldots \text{[b.1]}$$

the minimum absolute deviations estimator seeks the vector γ which minimizes the expression[1]

$$\sum |y - W\gamma| = \sum |u^t|$$

subject to

$$y = W\gamma + u^t$$

$$\gamma \geq 0 \quad \ldots\ldots\ldots\ldots\ldots \text{[b.2]}$$

Writing the u^t as the difference of two non-negative variables z^t and v^t

$$\sum |u^t| = \sum |v^t - z^t|$$

and thus the constraint set may be written

$$y = W \quad + [\, I \,,\, -I \,] \begin{bmatrix} v \\ z \end{bmatrix} \quad \ldots\ldots\ldots\ldots \text{[b.3]}$$

and hence to estimate any particular row of the k x k interregional growth operator for an n-year intercensal record we may use the simplex algorithm on the initial tableau

$$\begin{array}{cccc} & \gamma & z & v \\ y & W & I_n & -I_n \end{array} \quad \ldots\ldots\ldots\ldots\ldots \text{[b.4]}$$

---

[1]The derivation follows Rogers (56) but in the notation of this study.

APPENDIX C

PROGRAM LIBRARY DESCRIPTION AND MISCELLANEOUS RAW DATA LISTINGS

Copies of the source listings for all programs mentioned in this study are available from the writer, c/o Environmental Engineering, Department of Civil Engineering, University of Massachusetts, Amherst, Mass.01002, USA. All programs were written in CDC3600-FORTRAN, and tested and run on the CDC 3600 Computer of the University of Massachusetts Research Computing Center.

MCM. Program for the Monte Carlo Study of the Rogers Model with additive random errors. Summary flow chart is given on Figure 5. Sample output for typical runs is given on Tables C1 - C3.

MCMX. Program for the Monte Carlo Study of the Rogers Model with random variable growth operator elements (Chapter IV). Sample output is given on Table C4.

WALD. Program for estimating interregional growth operators by the modification to Wald's Method developed in Chapter III.

GRADP. Program for the decomposition of net migration into in- and out migration by the response surface algorithm developed in Chapter V. A summary flow chart is given on Figure 15.

POPB. Stochastic simulation population projection program using first-order autoregressive processes to generate birth, death and migration rates. Sample output is given on Figures 22 - 25, and a summary flow chart on Figure 19

POPC. Stochastic simulation population projection program for interregional systems. Will require extensive modification for application to regions other than the LPVRPD in view of the dependence of the projection on a set of explanatory variables that may vary from place to place. Summary flowchart is given on Figure 26 and output for one particular town in an multi-region set on Figure 27.

SIMGR. Residential location simulation program developed in Chapter IX. Requires 32K of memory core for a 40 x 40 grid. Output is given on Figures 36 and 37, and a summary flowchart on Figure 35.

SYSTEM IDENTIFICATION

$$\Gamma = \begin{bmatrix} 1.0044 & 0.0024 \\ 0.0132 & 1.0128 \end{bmatrix} \quad R_s=20:1 \quad N=25 \quad n=8$$

		ULS		MAD
$\sigma_u=1x10^{-4}$	$\hat{\Gamma}$ =	$\begin{bmatrix} 1.0044 & 0.0024 \\ 0.0107 & 1.0130 \end{bmatrix}$	W	$\begin{bmatrix} 1.0039 & 0.0025 \\ 0.0132 & 1.0128 \end{bmatrix}$
	$\hat{\Gamma}$ =	$\begin{bmatrix} 1.0046 & 0.0024 \\ 0.0140 & 1.0127 \end{bmatrix}$	UW	$\begin{bmatrix} 1.0043 & 0.0024 \\ 0.0188 & 1.0123 \end{bmatrix}$
$\sigma_u=1x10^{-3}$	$\hat{\Gamma}$ =	$\begin{bmatrix} 1.0070 & 0.0022 \\ 0.0146 & 1.0129 \end{bmatrix}$	W	$\begin{bmatrix} 0.9942 & 0.0034 \\ 0.1714 & 0.9985 \end{bmatrix}$
	$\hat{\Gamma}$ =	$\begin{bmatrix} 1.0058 & 0.0029 \\ -0.0067 & 1.0146 \end{bmatrix}$	UW	$\begin{bmatrix} 1.0036 & 0.0025 \\ 0.1510 & 1.0000 \end{bmatrix}$
$\sigma_u=1x1o^{-2}$	$\hat{\Gamma}$ =	$\begin{bmatrix} 0.9630 & 0.0062 \\ -0.0485 & 1.0168 \end{bmatrix}$	W	$\begin{bmatrix} 0.8169 & 0.0199 \\ 1.1379 & 0.9083 \end{bmatrix}$
	$\hat{\Gamma}$	$\begin{bmatrix} 0.9882 & 0.0039 \\ 0.6243 & 0.9573 \end{bmatrix}$	UW	$\begin{bmatrix} 0.8968 & 0.0123 \\ 1.4644 & 0.8799 \end{bmatrix}$

Table C1 : Monte Carlo Study, Rogers Model with additive random errors. Comparison of ULS and MAD growth operator estimates for an 8-year record, N=25.

Abbreviations used in Appendix Tables : UW = unweighted estimator
W = weighted estimator
ULS= unrestricted least squares
MAD= minimum absolute deviations

SYSTEM IDENTIFICATION	$\Gamma = \begin{bmatrix} 1.0200 & 0.0050 \\ 0.0120 & 1.0650 \end{bmatrix}$	R_s = 20:1 N=25 n=25			
		ULS		MAD	
$\sigma_u = 1\text{x}10^{-4}$	$\hat{\Gamma}$ =	$\begin{bmatrix} 1.0200 & 0.0050 \\ 0.0119 & 1.0650 \end{bmatrix}$	W	$\begin{bmatrix} 1.0197 & 0.0050 \\ 0.0103 & 1.0655 \end{bmatrix}$	
	$\hat{\Gamma}$ =	$\begin{bmatrix} 1.0200 & 0.0049 \\ 0.0120 & 1.0640 \end{bmatrix}$	UW	$\begin{bmatrix} 1.0200 & 0.0050 \\ 0.0120 & 1.0649 \end{bmatrix}$	
$\sigma_u = 1\text{x}10^{-3}$	$\hat{\Gamma}$ =	$\begin{bmatrix} 1.0195 & 0.0051 \\ 0.0144 & 1.0643 \end{bmatrix}$	W	$\begin{bmatrix} 1.0195 & 0.0051 \\ 0.0144 & 1.0643 \end{bmatrix}$	
	$\hat{\Gamma}$ =	$\begin{bmatrix} 1.0202 & 0.0049 \\ 0.0110 & 1.0652 \end{bmatrix}$	UW	$\begin{bmatrix} 1.0188 & 0.0053 \\ 0.0130 & 1.0647 \end{bmatrix}$	
$\sigma_u = 1\text{x}10^{-2}$	$\hat{\Gamma}$ =	$\begin{bmatrix} 1.0185 & 0.0053 \\ -0.0084 & 1.0710 \end{bmatrix}$	W	$\begin{bmatrix} 0.9727 & 0.0194 \\ 0.0868 & 1.0452 \end{bmatrix}$	
	$\hat{\Gamma}$ =	$\begin{bmatrix} 1.0069 & 0.0083 \\ 0.0210 & 1.0630 \end{bmatrix}$	UW	$\begin{bmatrix} 0.9938 & 0.0119 \\ 0.0721 & 1.0479 \end{bmatrix}$	

Table C2 : Monte Carlo Study, Rogers Model with additive random errors. Comparison of ULS and MAD growth operator estimates for a 15-year record, N=25

SYSTEM IDENTIFICATION			
$\Gamma = \begin{bmatrix} 1.0056 & 0.0030 \\ 0.0056 & 1.0193 \end{bmatrix}$ R_s = 20:1 N= 50 n=10			
σ_u	ULS		MAD
1x10^{-4}	$s^2(\hat{\hat{\Gamma}}) = \begin{bmatrix} 0.0011 & 0.0001 \\ 0.2304 & 0.0017 \end{bmatrix}$	W	$\begin{bmatrix} 0.0270 & 0.0002 \\ 0.9480 & 0.0008 \end{bmatrix}$
	$s^2(\hat{\Gamma}) = \begin{bmatrix} 0.0015 & 0.0001 \\ 0.1354 & 0.0016 \end{bmatrix}$	UW	$\begin{bmatrix} 0.0027 & 0.0002 \\ 0.1633 & 0.0014 \end{bmatrix}$
1x10^{-3}	$s^2(\hat{\Gamma}) = \begin{bmatrix} 0.12 & 0.0001 \\ 24.32 & 0.20 \end{bmatrix}$	W	$\begin{bmatrix} 0.19 & 0.0016 \\ 4.34 & 0.36 \end{bmatrix}$
	$s^2(\hat{\Gamma}) = \begin{bmatrix} 0.11 & 0.0009 \\ 24.32 & 0.20 \end{bmatrix}$	UW	$\begin{bmatrix} 0.12 & 0.0009 \\ 4.34 & 0.36 \end{bmatrix}$
1x10^{-2}	$s^2(\hat{\Gamma}) = \begin{bmatrix} 19.42 & 0.15 \\ 1524.0 & 12.6 \end{bmatrix}$	W	$\begin{bmatrix} 27.6 & 0.22 \\ 501.0 & 4.40 \end{bmatrix}$
	$s^2(\hat{\Gamma}) = \begin{bmatrix} 13.4 & 0.10 \\ 1670. & 13.5 \end{bmatrix}$	UW	$\begin{bmatrix} 7.27 & 0.05 \\ 1374.0 & 11.06 \end{bmatrix}$

Table C3 : Monte Carlo Study, Rogers Model with additive random errors. Comparison of standard deviation of estimated growth operators for various estimation modes.

SYSTEM IDENTIFICATION

$$\bar{\Gamma} = \begin{bmatrix} 1.0056 & 0.0030 \\ 0.0056 & 1.0193 \end{bmatrix} \quad R_s = 20:1 \quad N=100 \quad n=10 \quad \rho=0$$

	W	UW
$\sigma_u=1x1o^{-2}$ $\hat{\bar{\Gamma}} =$	$\begin{bmatrix} 1.0062 & 0.0029 \\ 0.0055 & 1.0193 \end{bmatrix}$	$\begin{bmatrix} 1.0058 & 0.0030 \\ 0.0055 & 1.0193 \end{bmatrix}$
$\sigma_u=5x10^{-2}$ $\hat{\bar{\Gamma}} =$	$\begin{bmatrix} 1.0058 & 0.0029 \\ 0.0056 & 1.0193 \end{bmatrix}$	$\begin{bmatrix} 1.0043 & 0.0031 \\ 0.0067 & 1.0193 \end{bmatrix}$
$\sigma_u=1x10^{-1}$ $\hat{\bar{\Gamma}} =$	$\begin{bmatrix} 0.9988 & 0.0036 \\ 0.0048 & 1.0193 \end{bmatrix}$	$\begin{bmatrix} 1.0064 & 0.0030 \\ 0.0046 & 1.0193 \end{bmatrix}$
$\sigma_u=2x10^{-1}$ $\hat{\bar{\Gamma}} =$	$\begin{bmatrix} 1.0077 & 0.0029 \\ -0.0011 & 1.0199 \end{bmatrix}$	$\begin{bmatrix} 0.9989 & 0.0036 \\ 0.0136 & 1.0186 \end{bmatrix}$
$\sigma_u=3x10^{-1}$ $\hat{\bar{\Gamma}} =$	$\begin{bmatrix} 0.9711 & 0.0059 \\ -0.0067 & 1.0200 \end{bmatrix}$	$\begin{bmatrix} 0.9853 & 0.0047 \\ 0.0047 & 1.0194 \end{bmatrix}$

Table C4 : Monte Carlo Study, Rogers Model with random variable operator elements. σ_u = standard deviation of the growth operator elements expressed as a fraction of the mean value of that element.

	BIRTHS				DEATHS			
	$\hat{\bar{\mu}}$	$\hat{\rho}_1$	$\hat{\rho}_2$	$\hat{\sigma}(\mu)$	$\hat{\bar{\delta}}$	$\hat{\rho}_1$	$\hat{\rho}_2$	$\hat{\sigma}(\delta)$
Springfield	23.3	0.62*	0.08	1.47	9.7	0.48	0.60	0.41
Holyoke	21.9	0.72*	-0.02	1.80	12.95	0.34*	0.26	0.66
Chicopee	24.6	0.96*	-0.2	1.76	7.9	0.53*	0.16	0.42
Agawam	19.4	0.67*	-0.12	2.0	7.3	0.22	0.38*	1.08
Amherst	19.9	0.57*	0.04	2.6	8.3	0.19	0.53*	1.44
Belchertown	13.4	0.35*	0.16	2.4	7.5	0.24	0.17	1.39
Easthampton	19.9	0.69*	-0.07	2.3	9.6	-0.08	0.07	1.04
East Longmeadow	17.8	0.52*	0.007	3.00	7.5	0.34	0.39	1.39
Granby	23.8	0.47*	0.26	4.43	6.8	0.22	0.11	2.4
Longmeadow	12.2	0.001	0.26	1.67	7.5	-0.17	-0.17	1.41
Ludlow	20.9	0.67*	-0.02	2.2	7.3	0.10	0.22	0.77
Northampton	17.5	0.72*	-0.001	1.35	10.2	0.32	-0.23	0.81
South Hadley	22.3	0.76*	0.01	2.0	8.3	0.13	0.27	1.1
Southwick	23.8	0.77*	-0.10	3.3	7.7	0.56*	0.002	2.29
West Springfield	21.6	0.61*	0.11	2.02	8.8	-0.08	-0.10	0.76
Westfield	21.2	0.78*	-0.16	1.76	10.4	-0.14	-0.37*	0.76
Wilbraham	17.9	0.25	-0.05	3.16	7.4	0.36*	0.03	1.47
Blandford	20.6	0.41*	0.08	4.98	11.3	0.03	-0.4*	4.22
Chester	18.6	-0.07	0.40*	4.46	12.7	0.21	0.11	3.9
Granville	19.9	0.38*	-0.04	6.7	12.2	0.39*	0.16	3.6
Hadley	21.1	0.25	0.08	3.9	8.5	0.31	0.001	2.3
Hampden	17.0	0.22	0.13	4.7	8.3	0.17	0.22	3.27
Huntington	20.7	0.21	0.04	5.1	13.1	-0.15	-0.35*	2.6
Middlefield	19.7	-0.04	0.08	8.1	10.1	0.06	0.13	5.6
Montgomery	19.1	0.08	-0.29	8.6	7.1	-0.05	-0.08	7.08
Pelham	19.2	0.04	0.11	5.9	9.7	0.02	0.29	4.4
Russel	22.3	-0.05	0.40*	4.5	10.2	-0.11	-0.02	2.7
Southampton	17.6	0.19	-0.11	3.2	8.3	0.16	0. 0*	2.9
Tolland	10.7	-0.1	0.17	10.7	9.7	0.10	0.13	11.2
Westhampton	24.3	0.14	0.04	7.3	9.9	-0.08	0.009	5.7
Hampden County	22.1	0.73*	-0.01	1.73	10.21	0.30*	0.067	0.32
Massachusetts	20.9	0.96*	-0.22	1.16	10.9	0.49*	0.22	0.30

Table C5 : Second-order autoregressive parameters for birth and death rates for communities in the LPVRPD over the interval 1940-1965

APPENDIX D

OUTLINE OF TRADITIONAL POPULATION PROJECTION METHODS

Least squares method. A least squares regression line is fitted to the historical record and extrapolated into the future.

Geometric. A least squares fit to the logarithm of historical values is computed and extrapolated to give the logarithm of future population.

Step-down. Assumes that trends in growth rates relative to larger regions remain constant. Least squares regression lines are generally employed to extrapolate the fraction of population in the study area.

Cohort survival. Based on the analysis of fertility, mortality and age structure (see Chapter VII). Excellent for closed population systems, but requires estimates of the numbers and age distribution of net migrants to be added at each time-step of the projection for open (local area) population systems.

The best review of traditional methods is contained in Isard (104), p. 1-80. McJunkin (18) surveys the treatment of the topic in the environmental engineering textbooks.

APPENDIX E : REFERENCES

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